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Certain Priorities Of The Generalized Mittag- Leffler Function

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Abstract
In the present paper, differentiation and integration of some generalized functions of Mittag-Leffler type are considered. Also derived the relations that exists between this function and the operators of Riemann-Liouville fractional integrals and derivatives. Some special cases of derived results are discussed. Results derived in this paper are the extensions of the results given earlier by Haubold et al.[7], Kilbas et al.[1] and Saxena et al.[11].
***Keywords**:- Fractional Calculus, Riemann-Liouville Fractional Integrals and Derivatives, M-Series, Generalized M-Series, K-Function.*

1- INTRODUCTION AND DEFINITIONS

The various Mittag-Leffler functions discussed in this paper will be useful for investigators in various disciplines of applied sciences and engineering. The importance of Mittag-Leffler functions in physics is steadily increasing. It is simply said that derivations of physical phenomena from exponential behavior could be governed by physical laws through Mittag-Leffler functions (power law. Currently more and more such phenomena are discovered and studied. It is particularly important for the disciplines of stochastic systems, dynamical systems theory and disordered systems. Eventually, it is believed that all these new research results will lead to the discovery of truly non-equilibrium statistical mechanics. This is statistical mechanics beyond Boltzmann and Gibbs. This non-equilibrium statistical mechanics will focus on entropy production, reaction, diffusion, reaction-diffusion,, and so forth, and may be governed by fractional calculus. Right now, fractional calculus and generalization of Mittag-Leffler functions are very important in research in physics.

The Mittag-Leffler function [5, 6]

$$E_\alpha(x) = \sum_{r=0}^{\infty} \frac{x^r}{\Gamma(\alpha r + 1)}, \quad (\alpha > 0) \tag{1.1}$$

and its generalized form[3]

$$E_{\alpha,\beta}(x) = \sum_{r=0}^{\infty} \frac{x^r}{\Gamma(\alpha r + \beta)}, \quad (\alpha, \beta > 0) \tag{1.2}$$

A generalization of (1.1) and (1.2) was introduced by Prabhakar [13] in terms of the series representation

$$E^{\gamma}_{\alpha,\beta}(x) = \sum_{r=0}^{\infty} \frac{(\gamma)_n x^r}{r!\Gamma(\alpha r + \beta)} \quad , (\alpha, \beta, \gamma \in C, \mathrm{Re}(\alpha) > 0) \tag{1.3}$$

Where $(\gamma)_n$ is Pochammer's symbol defined by

$$(\gamma)_n = \gamma(\gamma+1).......((\gamma+(n-1)), n \in N, \gamma \neq 0.$$

It is an entire function of order $\rho = [\mathrm{Re}(\alpha)]^{-1}$.

A generalization of (1.3) was defined by Sharma [9] as

$$ {}_pM_q^{\alpha}(a_1,\ldots,a_p; b_1,\ldots,b_q; x) = {}_pM_q^{\alpha}(x) = \sum_{r=0}^{\infty} \frac{(a_1)_r\ldots(a_p)_r}{(b_1)_r\ldots(b_q)_r} \frac{x^r}{\Gamma(\alpha r+1)} \quad (1.4)$$

Where $\alpha \in C, \mathrm{Re}(\alpha) > 0$ and $(a_j)_r$ and $(b_j)_r$ are the Pochammer symbols. The detailed information of this series is given in [9].

A generalization of (1.3) was defined by Sharma[10] as

$$ {}_pM_q^{\alpha,\beta}(a_1,\ldots,a_p; b_1,\ldots,b_q; x) = {}_pM_q^{\alpha,\beta}(x) = \sum_{r=0}^{\infty} \frac{(a_1)_r\ldots(a_p)_r}{(b_1)_r\ldots(b_q)_r} \frac{x^r}{\Gamma(\alpha r+\beta)} \quad (1.5)$$

Where $\alpha, \beta \in C, \mathrm{Re}(\alpha) > 0$ and $(a_j)_r$ and $(b_j)_r$ are the Pochammer symbols. The detailed information of this series is given in [10].

Recently, a new generalization of (1.3) was defined by Sharma [8] as

$$ {}_pK_q^{\alpha,\beta;\gamma}(a_1,\ldots,a_p; b_1,\ldots,b_q; x) = {}_pK_q^{\alpha,\beta;\gamma}(x) = \sum_{r=0}^{\infty} \frac{(a_1)_r\ldots(a_p)_r}{(b_1)_r\ldots(b_q)_r} \frac{(\gamma)_r x^r}{r!\Gamma(\alpha r+\beta)} \quad (1.6)$$

Where $\alpha, \beta, \gamma \in C, \mathrm{Re}(\alpha) > 0$ and $(a_j)_r$ and $(b_j)_r$ are the Pochammer symbols.The detailed information of this function is given in [8].

In the text, the following definitions [12] are needed:

- **Left-sided Riemann-Liouville fractional integral**

$$(I_{0+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \mathrm{Re}(\alpha) > 0. \quad (1.7)$$

- **Right-sided Riemann-Liouville fractional integral**

$$(I_{-}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1} f(t)dt, \mathrm{Re}(\alpha) > 0. \quad (1.8)$$

- **Left-sided Riemann-Liouville fractional derivative**

$$(D_{0+}^{\alpha} f)(x) = \frac{1}{\Gamma(1-\{\alpha\})} \left(\frac{d}{dx}\right)^{\alpha]+1} \int_0^x (x-t)^{\alpha-1} f(t)dt, \mathrm{Re}(\alpha) > 0.$$

$$= \left(\frac{d}{dx}\right)^{\alpha]+1} (I_{0+}^{1-\{\alpha\}} f)(x) \quad (1.9)$$

- **Right-sided Riemann-Liouville fractional derivative**

$$(D_{-}^{\alpha} f)(x) = \frac{1}{\Gamma(1-\{\alpha\})} \left(-\frac{d}{dx}\right)^{\alpha]+1} \int_x^{\infty} (t-x)^{-\{\alpha\}} f(t)dt, \mathrm{Re}(\alpha) > 0.$$

$$= \left(-\frac{d}{dx}\right)^{[\alpha]+1} (I_{-}^{1-\{\alpha\}} f)(x) \qquad (1.10)$$

Where $[\alpha]$ denotes the maximal integer not exceeding α and $\{\alpha\}$ is the fractional part of α.

2- RELATIONSHIP OF THE K-FUNCTION WITH ANOTHER SPECIAL FUNCTION

From the definition (1.6), we will get the following relations:

(i) If we set $\gamma = 1$, we get

$$_rK_s^{\alpha,\beta;1}(x) = {}_rM_s^{\alpha,\beta}(x) \qquad (2.1)$$

where $_rM_s^{\alpha,\beta}(x)$ is the generalized M-series introduced by Sharma and Jain[10].

(ii) If we take $\beta = \gamma = 1$, we arrive at

$$_rK_s^{\alpha,1;1}(x) = {}_rM_s^{\alpha,1}(x) = {}_rM_s^{\alpha}(x) \qquad (2.2)$$

where $_rM_s^{\alpha}(x)$ is the M-series given by Sharma[9].

(iii) If we put $r = s = 0$, we arrive at

$$_0K_0^{\alpha,\beta;\gamma}(x) = E_{\alpha,\beta}^{\gamma}(x) \qquad (2.3)$$

where $E_{\alpha,\beta}^{\gamma}(x)$ is the generalized Mittag-Leffler function given by Prabhakar[13].

(iv) If we put $r = s = 0, \gamma = 1$, we arrive at

$$_0K_0^{\alpha,\beta;1}(x) = E_{\alpha,\beta}^{1}(x) = E_{\alpha,\beta}(x) \qquad (2.4)$$

where $E_{\alpha,\beta}(x)$ is the generalized Mittag-Leffler function given by Wiman[3].

(v) If we put $r = s = 0, \beta = \gamma = 1$, we arrive at

$$_0K_0^{\alpha,1;1}(x) = E_{\alpha,1}^{1}(x) = E_{\alpha,1}(x) = E_{\alpha}(x) \qquad (2.5)$$

where $E_{\alpha}(x)$ is the Mittag-Leffler function given by Mittag-Leffler[5,6].

(vi) If we put $r = s = 0, \alpha = \beta = \gamma = 1$, we arrive at

$$_0K_0^{1,1;1}(x) = E_{1,1}^{1}(x) = E_{1,1}(x) = E_1(x) = e^x \qquad (2.6)$$

where e^x is the exponential function given in [5].

3- DIFFERENTIATION AND INTEGRATION OF THE K- FUNCTIONS

In this section differentiation and integration of the K-function (1.8) are presented in the forms of the theorem given below:

Theorem 3.1 Let $\alpha, \beta, \gamma, a \in C, \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta > n, \mathrm{Re}(\gamma), \mathrm{Re}(\delta) > 0$ then for $n \in N$ then there holds the relation:

$$\left(\frac{d}{dx}\right)^m \left\{x^{\beta-1} {}_rK_s^{\alpha,\beta;\gamma}(a\,x^{\alpha})\right\}(x) = x^{\beta-n-1} {}_rK_s^{\alpha,\beta-n;\gamma}(a\,x^{\alpha}) \qquad (3.1)$$

Proof:

In view of definitions (1.6), we get the desired result.

Theorem 3.2 Let $\alpha, \beta, \gamma, a, v, \sigma \in C, \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta > 0, \mathrm{Re}(\gamma), \mathrm{Re}(v) > 0, \mathrm{Re}(\sigma) > 0$ then there hold the relations:

$$\int_0^x (x-t)^{\beta-1} {}_rK_s^{\alpha,\beta;\gamma}[a(x-t)^{\alpha}]t^{v-1} {}_rK_s^{\alpha,v;\sigma}(a\,t^{\alpha})dt = x^{\beta+v-1} {}_rK_s^{\alpha,\beta+v;\gamma+\sigma}(a\,x^{\alpha}) \qquad (3.2)$$

Proof:
It can be made with the help of the Laplace transform formula

$$L\left\{t^{\rho-1}{}_rK_s^{\beta,\gamma;\delta}\left(a\,x^{\alpha}\right);s\right\}=\frac{s^{-\rho}\,\Gamma(\delta)\prod_{j=1}^{s}(b_j)_n}{\Gamma(\delta)\prod_{j=1}^{r}(a_j)_n}\times{}_{r+2}\Psi_{s+1}\left[\begin{matrix}(a_1,1),\ldots,(a_r,1),(\delta,1),(\rho,\alpha),(1,1)\\(b_1,1),\ldots,(b_s,1),(\gamma,\beta),(\delta,1)\end{matrix}\middle|\frac{a}{s^{\alpha}}\right],\qquad(3.3)$$

where $\mathrm{Re}(\beta)>0,\mathrm{Re}(\gamma),\mathrm{Re}(s)>0,\mathrm{Re}(\rho)>0,s>|a|^{1/\mathrm{Re}(\alpha)},\mathrm{Re}(\delta)>0.$

where ${}_{r+2}\Psi_{s+1}$ is the Wright function given in [7].

Remarks: If we set $r=s=0$ in above theorems, we get the results given by Kilbas et al. [1].

Special Cases

The theorems derived in the section 3 leads to the differentiation and integration of generalized M-series[10], M-series[9], generalized Mittag-Leffler function[3,6,13], Mittag-Leffler function[5,6] and exponential function[5] after implementing the necessary changes in the values of r,s,α,β and γ as mentioned in the section 2.

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