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Bounds on the Seidel Energy of Strongly Quotient Graphs

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ABSTRACT

The notion of Strongly Quotient Graph was introduced by Adiga, Liu and B. Liu defined the Seidel energy of a graph and its bounds. In this paper we obtain some S-eigen value, the upper and lower bounds on the Seidel energy of Strongly Quotient Graphs.

KEY WORDS: Seidel matrix, Seidel energy, Strongly Quotient graph.

1. INTRODUCTION

Let G be a connected simple undirected graph with n vertices and m edges. The vertices of G are labelled as $v_1, v_2, \ldots, v_n$ and referred as (n,m) graph. The Seidel matrix (Haemers, 2012; Liu, 2009) of a simple graph G with n vertices and m edges, denoted by $S(G) = (s_{ij})$, is a real symmetric square matrix of order n which is defined as

$$s_{ij} = \begin{cases} -1 & \text{if } v_i \text{ and are adjacent} \\ 1 & \text{if } v_i \text{and } v_j \text{ are not adjacent} \\ 0 & \text{otherwise.} \end{cases}$$

Also the Seidel matrix of the graph G, $S(G) = J - I - 2A$, where J denotes a square matrix whose entries are 1, I denotes an identity matrix and A is the adjacency matrix of the graph G. The eigen values of the Seidel matrix $S(G)$ are denoted by $s_1, s_2, \ldots, s_n$ and said to be S- eigenvalues of G.Since $S(G)$ is a real symmetric matrix, its eigenvalues are real numbers. The Seidel energy (Haemers, 2012; Liu, 2009) of the graph G, denoted by $SE(G)$, is defined *as* $SE(G) = \sum_{i=1}^{n} | s_i |$. Some lower and upper bounds for Seidel energy of connected and disconnected graph were obtained in (Nageswari, 2014).

During the past forty years or so, an enormous amount of research work has been done on graph labelling. A graph labelling is an assignment of integers to the vertices or edges or both, subject to certain conditions. These interesting problems have been motivated by practical problems. Recently, Adiga (2006), have introduced the notion of Strongly Quotient Graphs (Adiga, 2006; 2007; 2008; Zaferani, 2008). They derived an explicit formula for the maximum number of edges in a Strongly Quotient Graph of order n.

The labelling f of a graph G of order n is an injective mapping $f: V(G) \to \{ 1,2,\ldots,n \}$.Define the quotient function $f_q : E(G) \to Q$ by $f_q(e) = \min\left\{\frac{f(v)}{f(w)}, \frac{f(w)}{f(v)}\right\}$ if$e$ joins v and w. Note that for any $e \in E(G)$, $0 < f_q(e) < 1$.A graph with n vertices is called a Strongly Quotient Graph if its vertices can be labeled by$1, 2, \ldots, n$ such that the quotient function f_q is injective i.e., the values $f_q(e)$on the edges are all distinct. For more details on Strongly Quotient Graphs and to the properties of Strongly Quotient Graphsrefer (Adiga, 2006; 2007; 2008; Binthiya, 2014; Zaferani, 2008).

In this paper we obtain two eigenvalues of Seidel matrix of Strongly Quotient Graphs and present some bounds for the Seidel energy of Strongly Quotient Graphs.

Preliminaries: This section gives some lemmas which will be used in our main result.

Lemma 2.1 (Haemers, 2012; Liu, 2009). Let G be a connected graph of order n and let $s_1, s_2, \ldots, s_n$ be its S-eigenvalues. Then the S – eigenvalues satisfy the relations $\sum_{i=1}^{n} s_i = 0$ and $\sum_{i=1}^{n} {s_i}^2 = n(n-1)$.

Lemma 2.6 (Zhon. 2008) Let $a_1, a_2, \ldots, a_n$ be nonnegative numbers. Then

$$n\left[\frac{1}{n}\sum_{i=1}^{n} a_i - \left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}}\right] \le n\sum_{i=1}^{n} a_i - \left(\sum_{i=1}^{n}\sqrt{a_i}\right)^2 \le n(n-1)\left[\frac{1}{n}\sum_{i=1}^{n} a_i - \left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}}\right] .$$

Bounds on Seidel energy of Strongly Quotient Graph: Label the vertices $v_1, v_2, \ldots, v_n$ of the graph G order n such that $f(v_i) = i$ for $1 \le i \le n$, we obtained the strongly quotient graph with maximum possible edges m. Hence the vertices v_i and v_j are adjacent if gcd of i and j is 1 otherwise v_i and v_j are not adjacent. The notation (a,b) denotes the gcd of a and b.

Theorem 3.1: If G is a Strongly Quotient Graph with n vertices and maximum number of edges m then 1 is a S-eigenvalue of G with multiplicity greater than or equal to $| P |$, where $P = \{ p / p$ is a prime and $\frac{n}{2} < p \le n \}$.

Proof: Let p be any prime number such that $\frac{n}{2} < p \leq n$. The Seidel matrix for Strongly Quotient Graph with n vertices is

$$S(G) = \begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \\ R_4 \\ \cdot \\ \cdot \\ \cdot \\ R_n \end{array} \begin{array}{c} \begin{array}{cccccccccccc} C_1 & C_2 & C_3 & C_4 & . & . & . & C_p & . & . & . & C_n \end{array} \\ \left[\begin{array}{cccccccccccc} 0 & -1 & -1 & -1 & . & . & . & -1 & . & . & . & -1 \\ -1 & 0 & -1 & 1 & . & . & . & -1 & . & . & . & a_1 \\ -1 & -1 & 0 & -1 & . & . & . & -1 & . & . & . & a_2 \\ -1 & 1 & -1 & 0 & . & . & . & -1 & . & . & . & a_3 \\ . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . \\ -1 & -1 & -1 & -1 & . & . & . & 0 & . & . & . & -1 \end{array}\right] \end{array}$$

Where $a_i = s_{i\,n} = s_{n\,i} = \begin{cases} -1 & if \quad (n,i)=1 \\ 1 & if \quad (n,i) \neq 1 \end{cases}$ for i = 1, 2,…,(n-1).

So, the characteristic polynomial $\emptyset$ (G, λ) of S(G) is

$$\emptyset(G,\lambda) = \begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \\ R_4 \\ \cdot \\ \cdot \\ \cdot \\ R_p \\ \cdot \\ \cdot \\ \cdot \\ R_n \end{array} \begin{array}{c} \begin{array}{cccccccccccc} C_1 & C_2 & C_3 & C_4 & . & . & . & C_p & . & . & . & C_n \end{array} \\ \left|\begin{array}{cccccccccccc} \lambda & 1 & 1 & 1 & . & . & . & 1 & . & . & . & 1 \\ 1 & \lambda & 1 & -1 & . & . & . & 1 & . & . & . & -a_1 \\ 1 & 1 & \lambda & -1 & . & . & . & 1 & . & . & . & -a_2 \\ 1 & -1 & 1 & \lambda & . & . & . & 1 & . & . & . & -a_3 \\ . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . \\ 1 & 1 & 1 & 1 & . & . & . & \lambda & . & . & . & 1 \\ . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . \\ 1 & -a_1 & -a_2 & -a_3 & . & . & . & 1 & . & . & . & \lambda \end{array}\right| \end{array}$$

Replacing R_p by $R_p - R_1$, we get (λ + 1) is a factor of $\emptyset(G, \lambda)$. This is true for every $p \in P$. Therefore (λ - 1) $^{|P|}$ is a factor of $\emptyset$ (G, λ).Hence the proof.

Theorem 3.2: If G is a Strongly Quotient Graph, then -1 is a S-eigenvalue of G with multiplicity greater than or equal to β, where $\beta = \sum_{\substack{p-prime \\ p \leq \left\lfloor \frac{n}{2} \right\rfloor}} \lfloor \log_p n \rfloor$

Proof: If p is any prime number less than or equal to $\left\lfloor \frac{n}{2} \right\rfloor$, then the vertices v_p and v_{p^c} ($c = 2,3,\ldots,\lfloor log_p\, n \rfloor$)are not adjacent. If $j \neq p$ and $j \neq p^c$ then the adjacency between v_j and v_p is same as the adjacency between v_j and v_{p^c}.Hence

$$s_{ij} = \begin{cases} -1 & if\,(i,j)=1\, and\ i \neq j \\ 1 & if\ (i,j) \neq 1\, and\ i \neq j \\ o & otherwise \end{cases}$$ and

$$S(G) = \begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \\ \vdots \\ R_r \\ \vdots \\ R_{p^c} \\ \vdots \\ R_n \end{array} \begin{array}{c} \begin{array}{ccccccccc} C_1 & C_2 & C_3 & \cdots & C_p & \cdots & C_{p^c} & \cdots & C_n \end{array} \\ \left[\begin{array}{ccccccccc} 0 & -1 & -1 & \cdots & -1 & \cdots & -1 & \cdots & -1 \\ -1 & 0 & -1 & \cdots & -1 & \cdots & -1 & \cdots & a_2 \\ -1 & -1 & 0 & \cdots & -1 & \cdots & -1 & \cdots & a_3 \\ \vdots & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ -1 & -1 & -1 & \cdots & 0 & \cdots & 1 & \cdots & a_p \\ \vdots & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ -1 & -1 & -1 & \cdots & 1 & \cdots & 0 & \cdots & a_p \\ \vdots & \vdots & \vdots & & \vdots & & \vdots & & \\ -1 & a_2 & a_3 & \cdots & a_p & \cdots & a_p & \cdots & 0 \end{array}\right] \end{array}$$

Subtracting R_p from R_{p^c} in $\lambda I - S(G)$, it is given that $\lambda + 1$ is a factor of $\emptyset(G, \lambda)$ and this is true for $c = 2, 3, \ldots, \lfloor \log_p n \rfloor$. Then $(\lambda+1)^{\lfloor \log_p n \rfloor}$ is a factor of the characteristic equation, this is true for every $p \le \left\lfloor \frac{n}{2} \right\rfloor$. Hence the proof.

Theorem 3.3. Let G be a Strongly Quotient Graph with $n > 3$ vertices and maximum edges m. Let $P = \{p \,/\, p$ is a prime and $\frac{n}{2} < p \le n\}$ and $\alpha = |P|$. Then

$$SE(G) \le \alpha + \beta + \sqrt{(n - \alpha - \beta)\,(n(n-1) - \alpha - \beta)} \text{ ------------ (A)}$$

Proof: If $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ are real numbers then the Cauchy - Schwarz inequality states that

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \le \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right)$$

Setting $x_i = 1$ and $y_i = |s_i|$ andreplacing n by $(n - \alpha - \beta)$,we obtain

$$\left(\sum_{i=1}^{n-\alpha-\beta} |s_i|\right)^2 \le (n - \alpha - \beta)\left(\sum_{i=1}^{n-\alpha-\beta} |s_i|^2\right)$$

By Theorem 3.1 and Theorem 3.2, 1 and -1 are the S- eigenvalues of Strongly Quotient Graph with multiplicity greater than or equal to α and β respectively. Thus

$$(SE(G) - \alpha - \beta)^2 \le (n - \alpha - \beta)\,(n(n-1) - \alpha - \beta)$$

That is $SE(G) \le \alpha + \beta + \sqrt{n^2\,(n - \alpha - \beta - 1) + (\alpha + \beta)^2}$.

Theorem 3.4 Let G be a Strongly Quotient Graph with $n > 3$ vertices and maximum edges m. Let $P = \{p \,/\, p$ is a prime and $\frac{n}{2} < p \le n\}$ and $\alpha = |P|$. Then

$$SE(G) \le \alpha + \beta + \sqrt{\zeta + (n^2 - n - \alpha - \beta)(n - \alpha - \beta - 1)} \text{ ------------------(B)}$$

and $SE(G) \ge \alpha + \beta + \sqrt{(n-\alpha-\beta-1)(n+\alpha+\beta+\zeta) + (\alpha+\beta)^2}$

where $\zeta = (n - \alpha - \beta)\,|\det S(G)|^{2/(n-\alpha-\beta)}$

Proof: Lemma (2.6) states that

$$N \le n\sum_{i=1}^{n} a_i - \left(\sum_{i=1}^{n} \sqrt{a_i}\right)^2 \le n\,(n-1)\,N, \text{ where } N = n\left[\frac{1}{n}\sum_{i=1}^{n} a_i - \left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}}\right]$$

Replacing a_i by s_i^2 and n by $(n - \alpha - \beta)$

$$N \le (n - \alpha - \beta)\sum_{i=1}^{(n-\alpha-\beta)} s_i^2 - \left(\sum_{i=1}^{(n-\alpha-\beta)} |s_i|\right)^2 \le (n - \alpha - \beta - 1)\,N, \text{ ------------(1)}$$

where $N = \sum_{i=1}^{(n-\alpha-\beta)} s_i^2 - (n - \alpha - \beta)\left(\prod_{i=1}^{(n-\alpha-\beta)} s_i^2\right)^{\frac{1}{(n-\alpha-\beta)}}$.

By Theorem 3.1 and 3.2 , it is known that 1 and -1 are the S- eigenvalues of the Strongly Quotient Graph with multiplicity greater than or equal to α and β respectively.

Observe that

$$N = n^2 - n - \alpha - \beta - (n - \alpha - \beta)\,|\det S(G)|^{\frac{2}{(n-\alpha-\beta)}}$$
$$= n^2 - n - \alpha - \beta - \zeta$$

where $\zeta = (n - \alpha - \beta)\,|\det S(G)|^{\frac{2}{(n-\alpha-\beta)}}$

From (1) we get

$$[\,\mathrm{SE(G)} - \alpha - \beta)]^2 \geq (n-\alpha-\beta)(n^2-n-\alpha-\beta)-(n-\alpha-\beta-1)N$$

$$\Rightarrow \mathrm{SE(G)} \geq \alpha+\beta+\sqrt{(n-\alpha-\beta-1)(n+\alpha+\beta+\zeta)+(\alpha+\beta)^2}$$

and

$$[\,\mathrm{SE(G)} - \alpha - \beta)]^2 \leq (n-\alpha-\beta)(n^2-n-\alpha-\beta)-N$$

$$= \zeta+(n^2-n-\alpha-\beta)(n-\alpha-\beta-1)$$

$$\Rightarrow \mathrm{SE(G)} \leq \alpha+\beta+\sqrt{\zeta+(n^2-n-\alpha-\beta)(n-\alpha-\beta-1)}$$

Hence we get the upper and lower bounds for the Strongly Quotient Graph.

Note: Upper bound of (B) is sharper than upper bound of (A).

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