Article

Univalent Harmonic Functions

Meera Agarwal[1], Ranjita Kapur[2]

Abstract

The purpose of this article is to use the Dziok-Srivastava operator to find necessary and sufficient condition of complex valued harmonic univalent functions. Extreme points for these classes are also determined. An integral operator, distortion bounds and a neighbourhood of such functions are also considered.

Keywords: Convolution, Distortion bounds, Dziok-Srivastava operator, Harmonic functions, Integral operator, Neighbourhood

Introduction

Let O be the open unit disc and FH be the class of functions which are univalent, complex valued, sense-preserving, harmonic in O normalized by:

g (0) = gz (0) − 1 = 0.

Each g ϵ FH can be written as

g = h1 ± h2

Where h1 and h2 are analytic in O

We call h1 the analytic part and h2 be its co-analytic part. For g to be sense-preserving and locally univalent in O a necessary and sufficient condition is given by |h1' (z)| > |h2' (z)| in O.[1]

Thus for

g = h1 ± ϵ F, h2 H

We may write:

h1 (z) = z + $\sum^{\infty}$ = 2; h2 (z) = $\sum^{\infty}{}_{=1}$ (0 ≤v1 <1)

Note that FH becomes F, the class of normalized analytic functions which are univalent if the co-analytic part of is equal to zero.

For qt ϵ D (t = 1, 2 . . . p)

[1]Associate Professor, [2]Assistant Professor, Apeejay College of Fine Arts, Jalandhar, Punjab, India.

Correspondence: Ms. Meera Agarwal, Apeejay College of Fine Arts, Jalandhar, Punjab, India.

E-mail Id: meeraaggarwal17@yahoo.in

Orcid Id: http://orcid.org/0000-0002-8443-0942

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st ϵ D - {0,−1,−2 . . .} (t = 1, 2 . . . w), the generalized hyper geometric function is then defined by:

pFw (q1 . . . qp; s1 . . . sw; z)

= ∑ =0 ∞ (1) … ( ),

( ) …( ) ! 1

(p ≤w + 1; p, w ϵ N0 = {0, 1, 2 . . .})

Where (e) n is the Pochhammer symbol defined by:

( ) = Γ (+) Γ ( ) (+ 1) · (+ − 1) for n ϵ N = {1, 2 . . .} = 1 when n = 0.

Corresponding to the function:

h1 (q1 . . . qp; s1 . . . sw; z) = ZpFw (q1 . . . qp; s1. . . sw; z).

The Dziok-Srivastava operator:[2]

Ap, w (q1 . . . qp; s1 . . . sw) is defined by Ap, w (q1 . . . qp; s1 . . . sw) g (z) = h1 (q1 . . . qp; s1 . . . sw; z) ◻ g (z)

∞ (1) − 1 … ( ) − 1 = z + ∑ = 2 ( ) … ( ) −1 − 1! 1 − 1

Where "◻" means convolution

To make it easy, we write:

Ap, w [1] g (z) = Ap, w (q1 . . . qp; s1 . . . sw) g (z)

Now we define the Dziok - Srivastava operator of the Harmonic function given by:

g = h1 + h2

As + $A_{p,w}$ [1] h2

Ap, w [1] g = Ap, w [1] h1

Let F^*H (q1, s) be the family of harmonic functions of the form such that (arg Ap, w [1] g) ≥s, 0 ≤s <1, |z| = r <1.

For p = w + 1, q2 = s1 . . . qp = sw, F^*H (1, s) = FH (s)[3] is the class of orientation preserving univalent harmonic star like functions f of order s in O, that is (arg g (r) >s is univalent.[4]

Also, F^*H (n+1, s) = RH (n, harmonic functions class with (arg D^ng(z)) ≥s, where D is the Ruscheweyh derivative.[5]

We also let (q1, s) = F^*H (q1, s) ∩ VH[6], where VH, the class of harmonic functions g of the form (1) and there exists ◻ so that, mod 2, arg (uk) + (k – 1) ◻ = arg (vk) + (k − 1) ◻ = 0 k ≥2

Silverman and Jahangiri, gave the necessary and sufficient conditions for functions of the form (1) to be in (s), where 0 ≤s <1.[6]

Note for p = w+ 1, q1 = 1, q2 = s1 . . . qp = sw and the co-analytic part of g = h1 ± being equal to zero, the class (q s) reduces h2 - 1to the class studied in.[7]

Now here, we will present a sufficient condition for g = h ± given by (1) to be in F^* (q s) and then 1 h2 H1.

We will show that the same condition is also necessary for the functions to be in (q1, s). Distortion theorems, extreme points, integral operators and neighbourhoods of such functions are considered.

Theorems and Important Results

In theorem A, we will introduce a sufficient condition for the harmonic functions to be in F^*H (q1, s) theorem A.

Let g = h1 ± h2 be given by (1).

If $\sum^{\infty}_{=2}$ $_{-1!}^{1}$ ($_{1-s}$ $^{-s}$ | |+$_{1-s}$ $^{+s}$ | |) Γ (q1, s) ≤ 1- 1+s | $_1$| (5) 1−s

Where u1 = 1, 0 ≤ s < 1 and Γ (1) = |(1) −1 …( ) − 1 |, (1) −1 …( ) – 1 then g ϵ F^*H (q1, s)

Proof

In order to prove that g ϵ F*H (q1, s), we will show that if (5) holds, then the required condition (3) is satisfied.

For (3), we can write:

(arg Ap,w [1]g(z)) = Re {z(Ap,w[1]h1(z)) − (Ap,w[1]h2(z)) .}Ap, w[1]h1 + Ap,w[1]h2 = Re $^{( )}$ $_{( )}$

using the relation that Re ω ≥ s iff |1 − s + ω | ≥ |1 + s − ω |,

It is sufficient to show that:

|M(z) + (1−s)N(z)| − |M(z) − (1 + s)N(z)| ≥ 0. (6)

Substituting the values of M (z) and N (z) in (5), the expression becomes:

|M(z) + (1 − s)N(z)| − |M(z) − (1 + s)N(z)| ≥ (2 − s)|z| − $\sum^{\infty}_{=2}$ $^{+1-s}$ $_{-1!}$ Γ ($_1$,)| || | − $\sum^{\infty}_{=1}$ $^{-1+s}$ $_{-1!}$ Γ ($_1$,)| || | s|z| - $\sum^{\infty}_{=2}$ $^{-1-s}$ $_{-1!}$ Γ (1)| || | - $\sum_{=1}^{\infty}$ +1+s Γ(1)| || | (7)

1 − ∑∞ − Γ (,)| | ≥2(1−s)|z|{ =2 (1−)(−1)! 1 − $\sum^{\infty}$ + } Γ (,)| | $^{=1}$ (1−)(−1)! 1 = 2(1 − s)|z| {1 −

1+ s | | [$\sum^{\infty}$ 1 (k− s | | + 1 − s 1 − s 1 $^{=2}$ (−1)! k+ s | |) Γ (,)]} 1 − s 1

This expression is non-negative by (5) and so g ϵ F^*H (q1, s) we obtain the necessary and sufficient given by (4).

Conditions for g = h1 + h2

Theorem B

Let g = □□□ be given by (4). Then g ϵ □ (q1, s) h1 + h2

{ [∑$^{\infty}$ $_{=2}$ $_{(-1)!}$^{1} ($^{k-}$$_{1-s}$^{s} | | + $_{1}$$^{k+}$ $_{-}$ ^{s}$_{s}$ | |) Γ ($_{1,}$)]}≤ 1 − 1+ s |$_{1}$| (8) 1 − s

Where u1 = 1, 0 ≤ s < 1

$_{(1)}$ =| $^{(1)\,-1\,...(\,)}$ −1 $_{|}$ $^{(1)}$ −1 $^{...(\,)}$ −1

Proof

Since (q1, s) □ F^*H (q1, s), so we required to prove other part of the theorem.

For functions g ϵ (q1, s), we observe that the condition

(arg Ap, w[1]g) ≥s

(arg Ap, w[1]g) - s

= Re { z($A_{p,w}$[$_{1}$]h_1(z)) − ($A_{p,w}$[$_{1}$]h_2 (z))′ . − s}^{A}p,w$^{[}$ 1$^{]h}$1+ $A_{p,w}$[$_{1}$]h_2 ≥ 0

Re ∞ −s ∞ +s (1 − s)z + ∑ $_{=2}$ Γ ($_{1,}$)| | − $^{\Sigma}$ =1 −1! Γ ($_{1,}$)| | [−1!] ≥ ∞ Γ ($_{1}$) | | − ∞ 0 $^{+\Sigma}$ =2 $^{\Sigma}$ $_{=1}$ $^{\Gamma\,(}$ 1$^{,\,)}$ | | □ (9)

This condition holds for all values of z in O. if we choose □ according to (1.4), then we have:

(1 − s)– (1+ s)| $_{1}$|− ∑ $_{=2}$$^{\infty}$ (−s | |+ +s | |) Γ ($_{1,}$) $^{-1}$.

−1! −1! 1+| |+ ∑$^{\infty}$ (| |+ | |) Γ (,) $^{-1}$ ≥ 0 1 =2 1 (10)

If the given condition does not hold then the numerator in above equation is negative for r close to for k = 2, 3.

Therefore there exists a point z0 = r0 in interval (0, 1) for which the quotient of above equation is negative which is a contradiction and Hence the result.

Theorem C

Let suppose the values

□k = (1 − s) (k−1)! and □k = (1 − s) (k−1)! (−s)Γ (,) (+s)Γ (,) 1 1 (q1,

For v1 to be fixed, then the extreme points for (s) are given by:

{z + □kxz k̄ } where k ≥2∓1 }□ { z + 1 + □k (11) and |x| = 1 − |v1|.

Proof

Any function g ϵ □ (q1, s) may be expressed as:

g(z) = z + ∑∞ =2| |+̄ +̄ ∑ ∞ | | 1 = 2

Where the coefficients satisfy the inequality (5)

Set h11 (z) = z, h21 (z) = v1z, h1k (z) = z + □$_{k}$ $^{□}$

h2k = v1z +,

Writing Xk = | | □$_{k}$ Yk = | | k = 2, 3 . . . and X1 = 1 − ∑ $_{=2}$$^{\infty}$ Xk; Y1 = 1 − ∑ $_{=2}$$^{\infty}$ Yk

We have:

g(z) = ∑ $_{=1}$$^{\infty}$ ($X_k h_1$ ( ) + Y h_2 ( ))

In particular, we have

h21 (z) = z +̄1 and

h2k (z) = z + □kxz k̄ ∓1 +□ (k ≥2, |x| + |y| = 1 − |v1|),k

h2k (z) = z + □kxz k̄ ∓1 +□ (k ≥2, |x| + |y| = 1 − |v1|),k

We observe that the extreme points of □ (q1, s) are completely contained in {h2k(z)}.

To see that h21 is not an extreme point, Note that h21 may be written as

h21 (z) = ^{1}$_{2}${h21 (z) + □2 (1 − |v1|)z^2} + ^{1}$_{2}${ h21 (z) - □2 (1 − |v1|)z^2}, a convex linear combination of functions in □ (q1, s). If both |x| ≠0 and |y| ≠ 0, we will show that it can also be expressed as a convex linear combination of functions in □ (q1, s).

Wlog, assume |x| ≥ |y|. Choose > 0 small enough so that is strictly less than | | .

|y| Choose M = 1 + and N = 1 − | |.

Now we observe that both

t1(z) = z + □kMxz k̄ and +v1z + □k

t2(z)= z + □k(2-M)xz k̄ +v1z + □k (2 −) are in̄ (q1, s) and note that gn(z) = 12{t1(z) + t2(z)}.

This shows that such functions are the required extreme points for □ (q1, s) which proofs the theorem.

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