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Some Results Concerned To Fractional Logistic Equation And Generalized Mittag-Leffler Function

Kishan Sharma
Amity University Madhya Pradesh, Gwalior, India.
Email- drkishansharma2006@rediffmail.com

Abstract
In this paper, we obtain a solution of fractional logistic equation in terms of generalized Mittag-Leffler function.
***Keywords**: Generalized Mittag-Leffler function; Caputo fractional derivative; Logistic equation.*
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1- INTRODUCTION

In recent years applications of fractional calculus have been investigated extensively. Considerable amount of work has been done in area of fractional differential equations and many analytical and numerical methods were developed and employed for obtaining the solution reported by Mathai et al. [1]. It has been found that in some cases fractional calculus is more accurate than classical calculus to describe dynamic behavior of real world physical systems. Exponential function which arises by solving differential equation plays an important role for describing growth and decay in many physical applications. In fractional ordered differential equation, exponential function loses its properties to describe the solution and Mittag-Leffler function is used as its substitute.

A non linear differential equation of population growth model was first published by Verhulst [2], subsequently known as logistic equation,

$$\frac{du(t)}{dt} = ku(t)(1-u(t));\, t \geq 0; \tag{1}$$

whose exact closed form is given by

$$u(t) = \frac{u_0}{u_0 + (1-u_0)\, e^{-kt}} = ku(t)(1-u(t)); t \geq 0; \tag{2}$$

where u_0 is the initial state when time $t = 0$. This equation often arises while modeling ecology, neural networks, epidemics, Fermi distribution, economics, sociology etc. So, we are motivated to study fractional logistic equation by 5 generalizing (1) to its non-integer part.

In 2015, Bruce J. West [3] considered fractional form of non-linear logistic equation in following form,

$$D_t^{\alpha}[u(t)] = k^{\alpha} u(t)(1-u(t)) \tag{3}$$

He found solution using Carleman embedding technique as,

$$u(t)=\sum_{n=0}^{\infty}\left(\frac{u_0}{1-u_0}\right)^n E_{\alpha,\beta}[-n\,k^{\alpha}t^{\alpha}];t\geq 0; \qquad (4)$$

whereas in 2016, Area et al. [4], has shown that the solution given in (3) is not an exact solution of fractional logistic equation. Further, Ortigueira et al.[5] expressed exact solution of fractional logistic equation in terms of fractional Taylor series.

In this paper, we propose a new approach in light of Jumarie [6] concept to obtain solution of fractional logistic equation.

The Mittag-Leffler function introduced by MittagLeffler [7] in 1903 is defined as

$$E_{\alpha}(x)=\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(\alpha n+1)} \qquad (5)$$

Where $\alpha,\beta\in C,\mathrm{Re}(\alpha)>0.$

The generalization of (5) is given by Wiman[3]

$$E_{\alpha,\beta}(x)=\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(\alpha n+\beta)} \qquad (6)$$

where $\alpha,\beta\in C,\mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0.$

Due to direct involvement in generalization of ordinary differential equation to its non-integer order, Mittag-Leffler function is found very useful in many areas of science and engineering. Caputo's [8] definition of fractional derivative is given by

$${}_0^cD_t^{\alpha}u(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{u^n(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau \qquad (7)$$

2- MAIN RESULTS

Logarithm Function with generalized Mittag-Leffler Function as the base.

Let $E_{\alpha,\beta}:R\to R_+$ (one-one and onto, if exists) has inverse function $L_{\alpha,\beta}(\log_{E_{\alpha,\beta}})$,

we define $L_{\alpha,\beta}$ as

$$E_{\alpha,\beta}\{L_{\alpha,\beta}(y)\}=y \text{ where } y>0$$

$$L_{\alpha,\beta}\{E_{\alpha,\beta}(x)\}=x \text{ where } x\in R$$

$$E_{\alpha,\beta}(x)=y\ \ x\in R\ \ \text{E\_ (x) = y or }\ L_{\alpha,\beta}(y)=x$$

$$E_{\alpha,\beta}(x)\to\infty \text{ as } x\to\infty$$

$$E_{\alpha,\beta}(x)\to 0 \text{ as } x\to-\infty$$

$$L_{\alpha,\beta}(x)\to\infty \text{ as } x\to\infty$$

$$L_{\alpha,\beta}(x)\to-\infty \text{ as } x\to 0$$

(8)

This is important to note that $E_{\alpha,\beta}$ is not one to one in general and hence the log function of $E_{\alpha,\beta}$ is not always defined.

Proposition: Let $u = E_{\alpha,\beta}(x_1)$ and $v = E_{\alpha,\beta}(x_2)$, where $\alpha, \beta > 0$ with above conditions (8). Then,

(i) $E_{\alpha,\beta}(x_1 + x_2) = E_{\alpha,\beta}(x_1)\, E_{\alpha,\beta}(x_2)$

(ii) $\log_{E_{\alpha,\beta}}(u.v) = \log_{E_{\alpha,\beta}}(u) + \log_{E_{\alpha,\beta}}(v)$

(iii) $\log_{E_{\alpha,\beta}}(u \div v) = \log_{E_{\alpha,\beta}}(u) - \log_{E_{\alpha,\beta}}(v)$

We consider the following logistic fractional equation

$$\frac{d^\alpha u}{dt^\alpha} = k^\alpha u(1-u) \tag{9}$$

where $0 < \alpha, \beta \le 1$and $u = u(t)$.

We may write this as

$$\int \frac{d^\alpha u}{u} + \int \frac{d^\alpha u}{1-u} = k^\alpha \int dt^\alpha . \tag{10}$$

Jumarie [6] clarified that, some formulas do not hold for the classical Riemann-Liouville definition, but can be applied with the modified Riemann-Liouville definition.

Further using the proposition for obtaining the solution of (9),

$$\log_{E_{\alpha,\beta}} u - \log_{E_{\alpha,\beta}}(1-u) = \frac{k^\alpha}{\Gamma(2-\alpha)} \int t^{1-\alpha} dt^\alpha + C \tag{11}$$

where C is the integration constant. On using (8) and (11), we get

$$\frac{u(t)}{1-u(t)} = C\ E_{\alpha,\beta}\left[\frac{k^\alpha}{\Gamma(2-\alpha)} \int t^{1-\alpha} dt^\alpha\right] \tag{12}$$

where $0 < \alpha, \beta \le 1$ and on putting $\alpha = \beta = 1$ we get $\log_{E_1} x = \log_e x$. Here we also clarify that the numerical value of $u(t) \div (1-u(t))$ is approximately very close to $\frac{u(t)}{1-u(t)}$.

Initially, when time t = 0, we write, u(0) = u0 and C $= \frac{u_0}{1-u_0}$

Now substituting the value of C in (12), we arrive at

$$\frac{u(t)}{1-u(t)} = \frac{u_0}{1-u_0}\ E_{\alpha,\beta}\left[\frac{k^\alpha}{\Gamma(2-\alpha)} \int t^{1-\alpha} dt^\alpha\right] \tag{13}$$

On simplifying, we get

$$u(t)=\frac{1}{1+\frac{1-u_0}{u_0}\left\{E_{\alpha,\beta}\left[\frac{k^{\alpha}}{\Gamma(2-\alpha)}\int t^{1-\alpha}\,dt^{\alpha}\right]\right\}^{-1}},$$

(14)

Note: (i) If we set $\beta=1$ in above results, we obtain[12].

(ii) If we take $\alpha=\beta=1$, we arrive at [12].

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