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Fractional Calculus Operators In Electrical Engineering

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Abstract
In this paper, the authors obtain the solution of fractional differential equation associated with a LCR electrical circuit in a closed form in terms of the k-Mittag-Leffer function using Caputo fractional differential operators. The results derived in this paper are the extensions of the results given earlier by M.F. Ali et al[10].
Keywords*: RLC Electrical circuit, k-Mittag-Leffler function, Fractional derivatives.*

1- INTRODUCTION

The classical calculus was independently discovered in seventeenth century by Isaac Newton and Gottfried Wilhelm Leibnitz. The question raised by Leibnitz for the existence of fractional derivative of order, half was an ongoing topic amongst mathematicians for more than three hundred years, consequently several aspects of fractional calculus were developed and studied. During last decade applied mathematicians and physicists found the fractional calculus operators to be very useful in a variety of fields such as quantitative biology, electro chemistry, scattering theory, transport theory, probability, elasticity, control theory, potential theory, signal processing, image processing, diffusion theory, kinetic theory, heat transfer theory and circuit theory etc.. The fractional calculus operators also occur widely in technical problems associated with transmission lines and the theory of compressional shock waves. The fractional calculus is a generalization of ordinary differentiation to non-integer case. In other words, the fractional calculus operators deal with integrals and derivatives of arbitrary (i.e. real or complex) order. The name "fractional calculus" is actually a misnomer; the designation, "integration and differentiation of arbitrary order" is more appropriate. The first accurate use of a derivative of non-integer order is due to the French mathematician S. F. Lacroix [22] in 1819 who expressed the derivative of non-integer order $\frac{1}{2}$ in terms of Legendre's factorial symbol Γ. Starting, with a function $y = x, m$, Lacroix expressed it as follows Replacing with $\frac{1}{2}$ and putting $m = 1$, he obtained the derivative of order $\frac{1}{2}$ of the function x. The credit of first application of fractional calculus goes to Abel's [11] who employed it in the solution of an integral equation which emerged in the formulation of the tautochrone problem of finding the shape of a frictionless wire lying in a vertical plane such that the time of slide of a bead placed on the wire to the lowest point of the wire is the same regardless of position of the bead on the wire. The importance of special functions as a device of mathematical analysis is well known to the scientist, mathematician and engineers dealing with the practical applications of differential equations. The solution of various problems from the heat conduction, electromagnetic waves, Fluid mechanics, quantum mechanics, kinetic equations and diffusion equations etc. lead obligatory to using the special function. Special functions arise as a solution of some basic ordinary differential equations and solving partial differential equations by means of separation of variable method. The verity of the nature of the methods leading to special functions stimulated the increasing of the number of special functions used in applications.

The Mittag-Leffler function introduced by MittagLeffler [7] in 1903 is defined as

$$E_{\alpha}(x)=\sum_{n=0}^{\infty}\frac{x^{n}}{\Gamma(\alpha n+1)} \quad ....(1)$$

where $\alpha,\beta\in C,\operatorname{Re}(\alpha)>0.$

It's generalized form is given by Wiman[3]

$$E_{\alpha,\beta}(x)=\sum_{n=0}^{\infty}\frac{x^{n}}{\Gamma(\alpha n+\beta)} \quad ....(2)$$

where $\alpha,\beta\in C,\operatorname{Re}(\alpha)>0,\operatorname{Re}(\beta)>0.$

The generalization of the above functions is given by Prabhakar [23] in 1971 in the form

$$E_{\alpha,\beta}^{\gamma}(x)=\sum_{n=0}^{\infty}\frac{(\gamma)_{n}x^{n}}{n!\Gamma(\alpha n+\beta)} \quad ....(3)$$

where $\alpha,\beta,\gamma\in C,\operatorname{Re}(\alpha)>0,\operatorname{Re}(\beta)>0,\operatorname{Re}(\gamma)>0$ and $(\gamma)_{n}$ is the Pochhammer's symbol.

In 2012, G. A. Dorrego and R. A. Cerutti [6] introduced the generalization of (3) known as k-Mittag-Leffler function denoted by $E_{k,\alpha,\beta}^{\gamma}(x)$ and defined as

$$E_{k,\alpha,\beta}^{\gamma}(x)=\sum_{n=0}^{\infty}\frac{(\gamma)_{n,k}x^{n}}{n!\Gamma_{k}(\alpha n+\beta)} \quad ....(4)$$

where $(\gamma)_{n,k}$ is the k-Pochhammer's symbol. k-Pochhammer's symbol and k-Gamma function are given below

$$(\gamma)_{n,k}=x(x+k)(x+2k)......(x+(n-1)k),\gamma\in C,k\in R,n\in N \quad ....(5)$$

and

$$\Gamma_{k}(x)=\int_{0}^{\infty}e^{-t^{k}/k}t^{x-1}dt,k\in R,x\in C \quad ....(6)$$

Need more accurate convergence conditions. Particularly, $\Gamma_{k}(x)=\Gamma(x)$ as $k\rightarrow 1$.

Many numbers of definitions of fractional derivative are given by many mathematicians like Riemann-Liouville operator, Modified Riemann-Liouville fractional derivative, Caputo fractional derivative, Weyl Fractional operator, Tuan and Saigo Fractional Operators. The Riemann-Liouville fractional derivative of constant is not equal to the Caputo fractional derivative of constant viz the Caputo fractional derivative of constant is zero.The Laplace transform of $f(t)$ is denoted by $L\{f(t)\}$ and defined as

$$L\{f(t)\}=\int_{0}^{\infty}e^{-st}f(t)dt$$

Where s a parameter is may be real or complex. Its inverse is given by

$$L^{-1}\{\overline{f(s)}\} = f(t).$$

The Caputo fractional derivative of order $\alpha > 0$ is introduced by Caputo [9] in the form

$${}_a^c D_t^\alpha f(t) = I^{m-\alpha} D^m f(t) \; if \; m-1<\alpha\leq m, (\alpha) > 0, m \in N$$

Or

$${}_a^c D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^m(t) dt$$

$$= \frac{d^m f(t)}{dt^m} \quad \text{if } \alpha = m .$$

Where $\frac{d^m f(t)}{dt^m}$ is the m-th derivative of the function (t) with respect to t.

If (t) is constant then ${}_a^c D_t^\alpha f(t) = 0$.

That is, Caputo's fractional derivative of a constant is zero.

The Caputo fractional derivative is a short of regularization in the time origin for the Riemann-Liouville fractional derivative.

The Laplace transform of Caputo derivative is representation of

$$L\{{}_a^c D_t^\alpha f(t)\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{k-1}(0) \quad \text{.... (7)}$$

When the initial conditions are zero then the equation (11) reduces to

$$L\{{}_a^c D_t^\alpha f(t)\} = s^\alpha F(s)$$

The importance of the Mittag-Leffler function and its generalizations due to their applications in several fields of science and engineering. The applications of the Mittag-Leffler functions are observed recently in a number of papers, related to fractional calculus and fractional order differential and integral equations and systems. Soubhia, Camargo and Rubens [2] have derived some applications of the Mittag-Leffler function in electrical engineering.

Let (t) be the ramp function. The ramp mathematically expressed as follows:

$$f(t) = \begin{cases} 0 \; for t < 0 \\ t \; for t \geq 0 \end{cases} \quad \text{... (8)}$$

and its Laplace transform is s^{-2}.

Let (t) be the parabolic function. The Parabolic mathematically expressed as follows:

$$f(t) = \begin{cases} 0 \quad for t < 0 \\ t^2/2 \; for t \geq 0 \end{cases} \quad \text{... (9)}$$

and its Laplace transform is s^{-3}.

2- LCR – ELECTRICAL CIRCUIT

In this paper, we present RLC electrical circuit with a capacitor and an inductor are connected in parallel and this set is connected in series with a resistor and voltage. The capacitance C, the inductance L and the resistor R are consider positive constants and (t) is the ramp function [2]. Consider the (t) is Heaviside function. The constitutive equations associated with a three elements of *RLC* electrical circuit are:

The voltage drop $U_L(t) = L\frac{d}{dt} I(t)$, across an inductor;

The voltage drop

$$U_R(t) = RI(t), \text{ across a resistor;}$$

The voltage drop

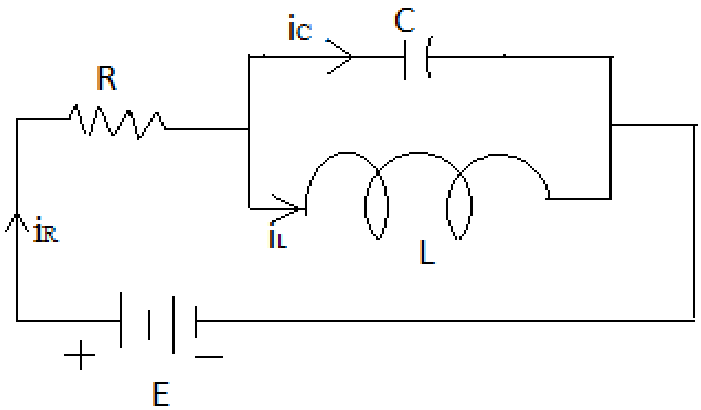
$$U_C(t) = \int_0^t I(\zeta)d\zeta, \text{ across a capacitor}$$

where $I(t)$ is the current.

Applying the Kirchhoff's voltage law and constitutive equations associated with the three elements, we can write the non-homogeneous second order ordinary differential equation

$$RC\frac{d^2}{dt^2}U_C(t) + \frac{d}{dt}U_C(t) + \frac{R}{L}U_C(t) = \frac{d}{dt}\Psi(t) \qquad \text{....(10)}$$

where $Uc(t)$ is the voltage on the capacitor, this is the same on the inductor as we can see in figure 1, because they are connected in parallel.



Three element LCR electrical circuit

Fig-1

On the other hand, we obtain other non-homogeneous second order ordinary differential equations associated with the current on the inductor,

$$RLC\frac{d^2}{dt^2}i_L(t) + L\frac{d}{dt}i_L(t) + Ri_L(t)) = \Psi(t) \qquad \text{.....(11)}$$

Again, using the constitutive equation for the inductor, these two non-homogeneous second order ordinary differential equations can be led to correspondent integro-differential equations,

$$R\frac{d}{dt}i_C(t) + \frac{1}{c}i_C(t) + \frac{R}{LC}\int_0^t i_C(\zeta)d\zeta = \frac{d}{dt}\Psi(t) \qquad \text{....(12)}$$

and

$$RC\frac{d}{dt}U_L(t) + U_L(t) + \frac{R}{L}\int_0^t U_L(\zeta)\, d\zeta = \Psi(t) \qquad \text{.....(13)}$$

respectively. We note that, integro-differential equations have the some form. Here we consider only the first one. The classical methodology to discuss this integro-differential equation is the Laplace transform. To this end, we consider the initial condition $i_C(0) = 0$ and the solution can be found in terms of an exponential function [8].

3- FRACTIONAL INTEGRO-DIFFERENTIAL EQUATION

In this section we discuss the fractional form of equation (12), i.e. a Fractional integro-differential equation associated with a current on the capacitor,

$$R\frac{d^{\alpha}}{dt^{\alpha}}i_C(t)+\frac{1}{C}i_C(t)\ +\frac{R}{LC}\frac{1}{\Gamma(\alpha)}\int_0^t\ (t-\zeta)^{\alpha-1}i_C(\zeta)d\zeta\ =\frac{d}{dt}\Psi(t) \qquad \text{....(14)}$$

We also consider $i_C(0)=0$, i.e., the initial current on the capacitor is zero. We note that this equation is a possible generalization of the classical integro-differential equation 44 M. Ali et al.: An application of fractional calculus associated with the *RLC* electrical circuit, because for α = 1 we recover the results obtained in (12). This replacement can be useful in discussing the corresponding numerical problem, for a particular value of the parameter, because the solution is presented in terms of a closed expression.

To solve this fractional integro-differential equation, we introduce the Laplace integral transform, defined by $L\{i_C(t)\}=F(s)=\int_0^{\infty}\ e^{-st}i_C(t)dt$

With $R(s)>0$, and we obtain the following algebraic equation

$$Rs^{\alpha}F(s)+\frac{F(s)}{C}+\frac{R}{LC}\frac{F(s)}{s^{\alpha}}=\frac{1}{s}$$

Whose solution is given by

$$F(s)=\frac{1}{R}\frac{s^{\alpha-1}}{s^{2\alpha}+as^{\alpha}+b}$$

Where we have introduced the positive parameters $a=\frac{1}{RC}$ and $b=\frac{1}{LC}$.

To recover the solution of the fractional integro-differential equation, we proceed with the inverse Laplace transform

$$i_C(t)F(s)=\frac{1}{R}L^{-1}\left\{\frac{s^{\alpha-1}}{s^{2\alpha}+as^{\alpha}+b}\right\}$$

Using the relation [12]

$$L^{-1}\left\{\frac{s^{\rho-1}}{s^{\alpha}+As^{\beta}+B}\right\}=s^{\alpha-\rho}\sum_{r=0}^{\infty}\ (-A)^r t^{(\alpha-\beta)r}E_{k,\alpha,\alpha+1-\rho+(\alpha-\beta)r}^{r+1}\quad(-Bt^{\alpha})$$

valid for $\left|\frac{s^{\beta}}{s^{\alpha}+b}\right|<1$ and $\alpha\geq\beta$.

We can write,

$$i_C(t)=\frac{t^{\beta}}{R}\sum_{r=0}^{\infty}\ (-a)^r t^{\alpha r}E_{k,2\alpha,\alpha+1+\alpha r}^{r+1}(-bt^{2\alpha})\ \psi(t) \qquad \text{.....(15)}$$

Where $E_{k,2\alpha,\alpha+1+\alpha r}^{r+1}(-bt^{2\alpha})$ is the generalized Mittag-Leffler functions and (t) is the Ramp function.

Again, if we consider (t) as a parabolic function then the solution will be

$$i_C(t)=\frac{t^{\alpha+1}}{R}\sum_{r=0}^{\infty}\ (-a)^r t^{\alpha r}E_{k,2\alpha,\alpha+2+\alpha r}^{r+1}(-bt^{2\alpha})\ \psi(t) \qquad \text{...... (16)}$$

4- SPECIAL CASES

In this paper we obtain new results in terms of k- Mittag- Leffer function. If we set k = 1 in the main result we arrive at the results given by [10].

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