Pisco Med Publishing

The Geometry of Gothic Architecture and the Proportions of the Music of the Spheres

Josep Lluis i Ginovart, Mónica López-Piquer

School of Architecture Barcelona, Universitat Internacional de Catalunya

Abstract: The heptagonal shape and its geometric layout have been the subject of a great deal of speculation. Because some apses in Gothic cathedrals are heptagonal, there must be a methodology implicit in the layout of the geometric shape. Two particularly important sources help us arrive at an understanding: the exceptional of the Capitular archive of the Cathedral of Tortosa, which contains the main neo-Platonic sources among its codices dating from thirteenth and fourteenth centuries, and the parchment known as la traça de Guarc (c.1345-1380), which shows the layout of the non-constructed cathedral. These sources show a heptagonal apse with an arithmetical and geometric dimension, based on a metrological and tonal musical proportion of 9/8, which is perfectly compatible with the bases of the quadrivium. The lateral and radial chapel, as the basic unit and feature element in fourteenth-century Gothic cathedral design, can be used as a pattern, and its measurement establish the basic unit for the overall proportions of the cathedral. This is the Music of Spheres that also appears at the Harmonices Mundi, Livri V (1619), by Johannes Kepler.

Keywords: Gothic; Medieval Geometry; Proportions; Music of Spheres; Gothic Cathedral

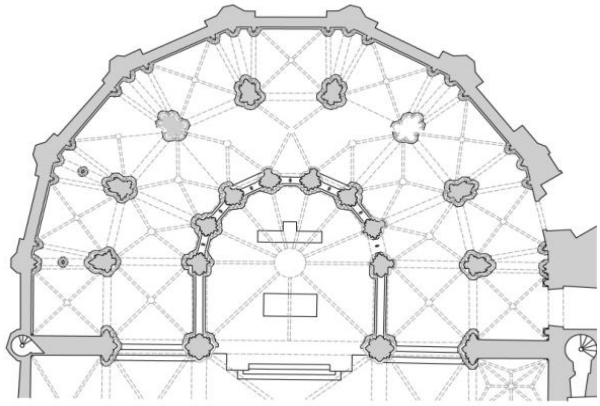
1. Introduction

Studies on the construction of the apse of the Gothic cathedral of Tortosa, Spain (1383-1435), relating the knowledge of the promoter and the master mason, allow a wide-ranging approach to the mathematical and geometrical knowledge in medieval architecture. This cosmology would last almost until the *Harmonices Mundi, Libri V* (1619), by Johannes Kepler (1571-1630), where the world is shown to be governed by the music of the spheres. It is an order based on the concept of proportion as mentioned in Plato's Timaeus (ca. 427-347 a.C), the base of which could qualify as being mathematic in modern terminology. Several studies have been made on different aspects of the Tortosa cathedral, e.g. the study of medieval design through an analysis of Guarc's scroll (ca. 1345-1380): *Gothic construction and the traça of a heptagonal apse. The problem of the heptagon* (Lluis i Ginovart, J. et. alt., 2013); a paper which shows the importance of architectural proportions when constructing the cathedral's apse: *Design and medieval construction: The case of Tortosa cathedral* (1345-1441) (Lluis i Ginovart; Costa, 2014) and also a study on the mathematical knowledge which clergymen may have contributed to the construction of the cathedral: The sciencia theorica speculativa in gothic construction. The church promoter of the medieval cathedral. This short summary provides a review on the proportions of Gothic cathedrals and, indirectly, on the mathematical knowledge behind them. The wide-ranging analysis will conclude that the radial chapel is the fundamental unit used to calculate the apse's measures, and also this unit is the edge length of a 14-sided regular polygon.

doi: 10.18282/adr.v1i2.550

1.1 The role of geometry in the construction of sacred space

Gothic architecture adapted spaces, especially apses, to the new liturgy brought by the *Prochiron*, *vulgar rationale divinorum officiorum* (1291), by *Guillaume Durand* (1230-1296), leading to the creation of semicircular walkways. This new vision comes to replaces the allegorical vision described in Gemma animae (ca. 1120) by Honorius of Autun (1080-ca. 1153), which is typical of the Romanesque style and translates into a linear apse and apsidioles. This would bring a major change in the conception of space, thus encouraging the creation of semicircular apses which open into five or seven surrounding radial chapels. As a consequence, the medieval master builder had to know a method for inscribing a 10 or 14-sided polygon in a circle; or in other words, how to draw a pentagon and a heptagon While the apses with a pentagonal base (having ten sides) can be constructed according to the principles stated in Book IV (P. 11-14) of Euclid's *Elementa*, translated by Adelard of Bath (1075-1166) around 1142, and Book I (P. 9) of Ptolemy's *Almagest* (ca. 85-165), translated by Gerard of Cremona (1114-1187) around 1175, designs with an heptagonal base could not be built because they are not described in those texts. Besides, the measure of this chapel was the fundamental unit used to construct the apse of Tortosa cathedral. (**Figure 1**).



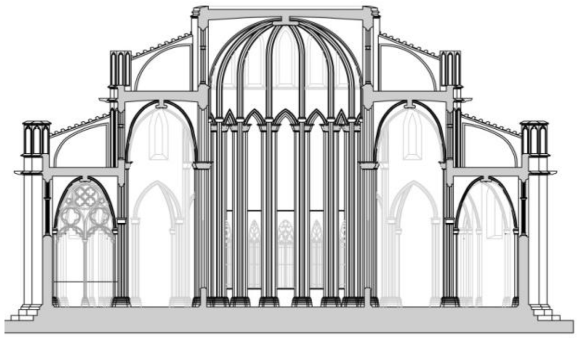


Figure 1; Plan and cross-section of the apse of Tortosa Cathedral (1383-1435).

2. Medieval design through Guarc's scroll

There are few medieval documents explaining a scientific method to construct regular polygons. One of the best known is the Par chu portrait om one toor a chinc arestes, by Villard de Honnecourt (ca. 1175-1240), (BnF. ms.fr. 190093, fol. 20 r c) (Lluis i Ginovart, J., 2014), where the pentagon is constructed with a set square (Sarrade, 1986).

In the chapter house archive of Tortosa cahedral (ACTo) there is a scroll (917x682 mm) showing the plan of a cathedral, with the annotation "En Antony Guarç". This document provides insight into the layout aids (lines and compass center points) used to draw the final plan. The initial measure of the grid in which the plan was laid out is 91 mm (account unit). The width of the cathedral is divided in parts measuring 91, 91, 182, 91 and 91 mm, corresponding to the three aisles. Each side chapel is 82 mm wide and the separation wall between chapels is 9 mm wide, so the ratio is 8/1. Therefore, the ratio of the side aisle to the radial chapel is 91/82. (**Figure 2**).

In order to lay out the seven chapels on the arc which defines the apse's semicircle, Guarç transfers the measure of the radial chapel (8 modules) to the chancel diameter (18 modules). All in all, he is using the numerical ratio 18/8 between the central aisle and the side chapel, i.e. ratio 9/8 between the walkway's width and the radial chapel's width. From the relationship between the radius, the 18 modules of the walkway's semicircle and the 8 modules of the radial chapel, we obtain a geometric and metrological solution.

According to the theory of proportions, if the chancel is 18 modules in width, the radial chapels must be 8 modules in width. In the apse, in order to construct a chapel measuring 3 canas (24 spans), we need a radius of 6 canas and 6 spans (54 spans).

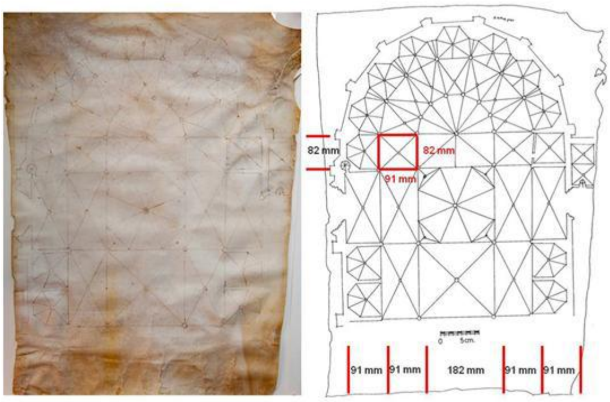


Figure 2; Guarç's layout (ca. 1345-1380) ACTo, Fabrica 49**.**

2. The construction's proportionality

The Gothic cathedral of Tortosa features an apse with a double walkway, having seven radial chapels on the chancel's semicircle. The apse was built between 1383 and 1435 in accordance with Durand's liturgy, and came to substitute the Romanesque cathedral which had been consecrated in 1178. The first phase was the belt of radial chapels, which were built one by one between 1383-1424. These chapels were laid out around the Romanesque cathedral. Therefore, the medieval master builder had to know a method for dividing a semicircle into seven equal parts, and he also had to lay out an heptagon without using the center point of the circumference, since it was already occupied by the Romanesque cathedral. (**Figure 3**).

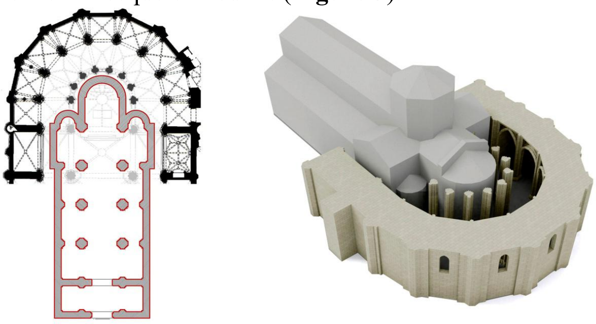


Figure 3; The Gothic cathedral on top of the Romanesque cathedral. J. Lluis, A. Costa, A. Ferré**.**

Three topographic surveys have been made in the Tortosa cathedral using the common techniques: tachymetry (2000), photogrammetry and tachymetry (2012), and laser scanning (2013). With the results thus achieved, which enable us to analyse the measurements and the metrology used in the construction of the apse, we have been able to determine that the basic unit of measure for the Tortosa cathedral is the 8-span cana. These units of measure are defined in volume IX, heading 15.5, of the book Consuetudines Dertosae (1272). The cana is 185,8 cm in length, and the span is 23,23 cm. If we measure the nine radial chapels of Tortosa cathedral from the geometric centers of the pillars, the following results are obtained:

Chapel measures		
Chapel number	Measure (meters)	Measure (canas)
1	5.59	3.0080
2	5.47	2.9434
3	5.64	3.0349
4	5.62	3.0241
5	5.52	2.9703
6	5.65	3.0349
7	5.64	3.0349
8	5.44	2.9272
9	5.68	3.0564
Total		27.0340
Average		3.0038
Cana	1.8584	

Therefore, the average width of the nine chapels is 3.0038 canas. If we only analyse the mean of the seven chapels which are located on the semicircle, we obtain a value of 2.9957 canas, measured from the geometric centers of the pillars. Therefore, from a metrological point of view, we can establish that the radial chapels of the cathedral have a measure of 3 canas, which are 24 spans.

The points corresponding to the geometric centers of the pillars of the apse are staked out 12.44 m away from the center of the chancel, i.e. a distance of 53.5514 spans, which is a 0.8376% error with regard to the metrological measure of 54 spans. Thus, there is a relationship between the circle radius (54 spans) and the edge length of the 14-sided polygon (24 spans), or in other words, a relationship between the radius where the radial chapels are laid out (18 modules) and the chapels measure (8 modules).

3. Mathematical knowledge of the promoter

In the chapter house archive some sources have been identified which were available to the clergymen and deal with numerical proportionality.

Amongst the codices which deal with mathematical topics, we must point out Saint Augustine's Civitatis Dei (354-420), ACTo n º 20; Calcidius' translation of Plato's Timaeus (f.350), ACTo no 80, (Figura 4); a part of Calcidius' commentary on the Timaeus (f.350), ACTo 236 (fol.39); an excerpt from Martianus Capella's De Nuptiis Philologiae et Mercurii (f. 430), vol.VII, Geometry, ACTo 80; a rendition by Al-Ḥajjāj ibn Yūsuf ibn Maṭar (786-833) on Euclid's Elementa (c.325-c.265 b.C.), ACTo 80; an excerpt from Gerbert of Aurillac's (c. 940-1003) Geometria Incerti Auctoris, ACTo 80; the volume on position-based numerals ACTo 80 (fol.162r.1-3) written by someone close to Adelard of Bath (1090-1160); and, finally, Macrobius' Comentarii In Somnium Scipionis, ACTo 236 (f.400) (**Figure 5**).

In his codex, Martianus Capella (ACTo 80) proposes two kinds of lines: rhētós and álogos. In the layout of the Gothic heptagon, the chapels belonging to the straight section of the apse and those located on the apse chord have the same size and bear the same proportion to the radius. Using Capella's terminology, the measure of the chapel is a rhētós line. Furthermore, Guarç's ratio (9/8) is understood by Calcidius ACTo 80 and by Marcobius ACTo 236 as the whole and eighth (1+1/8), which is called epogdous. Therefore, the used ratio 18/8 (9/8) was well known to the promoters who had the cathedral built.

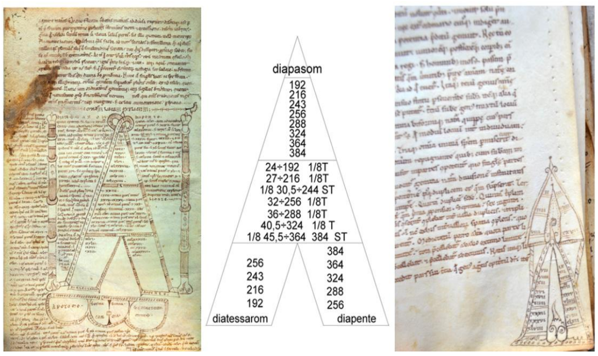


Figure 4; Calcidius' commentary on the Timaeus (ACTo no. 80).

In the catalogue of the chapter house archive ACTo 300 we find the Tractatus varii de rebus philosophicis et mathematicis. Ambianis (1509-1510) (Bayerri, 1962). This is a collection of philosophical and mathematical texts by Charles Bouvelles (1478-1567) which were published in Paris in 1510. Bouvelles is the author of the first Geometrie en francoys. Cy commence le livre de l'art et science de geometrie avecques les figures sur chacune rigle (1511), later reissued as Livre singularis touchant l'art et pratique de Geometrie (1542).

Outside the knowledge of the gothic lodges, Bouvelles would publish one of the first solutions to the construction of the heptagon in Libellus de mathematicis supplementis (1509), (Mas. Sup. fol.192). This solution differs considerably from the 18/8 solution proposed by Guarç.

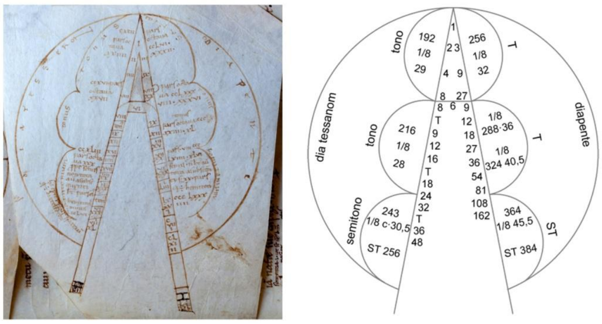


Figure 5; Comentarii In Somnium Scipionis, by Macrobius (ACTo. 236).

4. The construction of the heptagon and the medieval order

In the books of practical geometry which date from the Gothic period, the heptagon is constructed by means of a compass. That is how it is done in M. Roritzer's Geometrie Deutsch (1488) (fol. 2v.), the first of these books to be published and spread as know-how of medieval master builders. Albrecht Dürer (1471-1528) did the same in Underweysung der Messung, mit dem Zirckel und Richtscheyt: in Linien Ebnen vo gantzen Corporen (1525), through his corollary on the construction of a pentagon (LII.15) and a heptagon (LII.11) (Dürer, 1525). These constructions have their origin in the books of the Arab mathematician Abū al-Wafā Al-Būzjānī (940-998), both his Geometry: Kitāb fī mā yaḥtāju al-ṣāni‘ min al-a‘māl al-handasiyya (Book on those geometric constructions which are necessary for craftsmen) (c.993-1008) (woepcke, 1855), and his Arithmetic: Kitāb fī mā yaḥtaj ilayh al-kuttāb wa’l-ʿummāl min ʾilm al-ḥisāb (Saidan, 1974) (**Figure 6**). These methods use a ruler and a compass and therefore they provide a geometric solution, not an arithmetic solution.

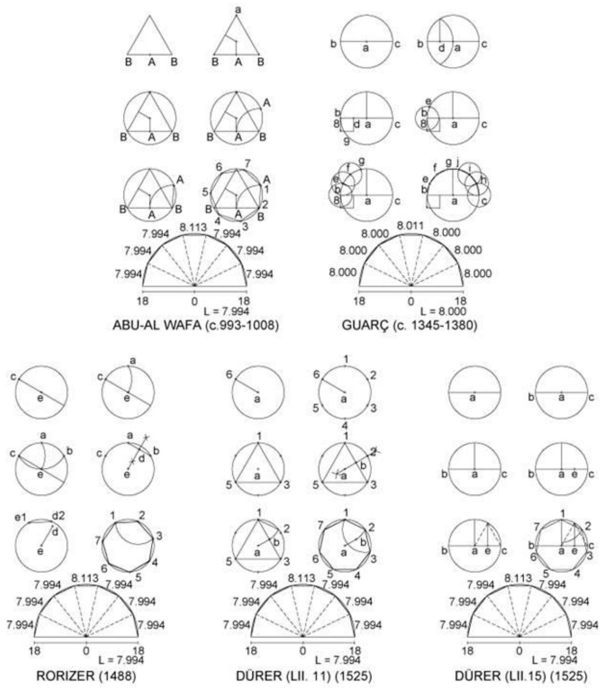


Figure 6; Geometric constructions of heptagons.

The construction of the heptagon using a ruler and a compass was later rejected by J. Kepler (1571-1630), in his Propositio. Heptagonus et figurae ab eo ommes (LI.45) (Kepler, 1864), included in Harmonices Mundi, Libri V (1619), and also by C. F. Gauss (1777-1855), in his Disquisitiones Arithmeticae (1801), (Section VII - Statements 361-366) (Gauss, 1801).

5. Implementation of medieval geometry in the construction of the cathedral

Guarç's ratio (9/8), which is used in the apse layout and is a tonal modulation, is not mentioned in any scholarly treaties on architecture or mathematics. Nonetheless, from a statistical point of view, the layouts made with these integers (9 and 8) are more exact than the layouts made with a compass, derived from the methods of Abū al-Wafā. This 9/8 ratio for the construction of the heptagon is used in the geometria fabrorum. This solution makes it possible to connect a geometric solution and an arithmetic solution. Tortosa cathedral is built using the measure of the chapel (3 canas, ie. 24 spans) (**Figure 7**).

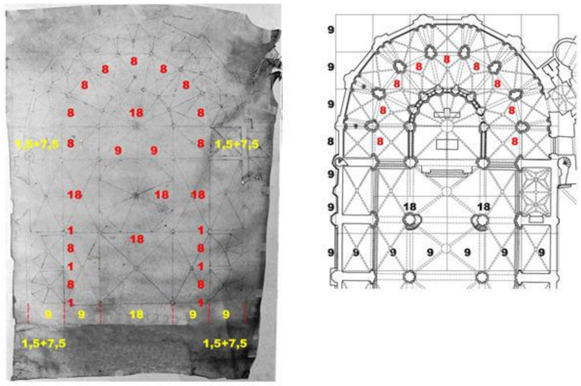


Figure 7; Guarc's layout (c. 1345-1380) and staking out of the apse (1383-1435).

There is a cosmological order where all elements are related and the fundamental unit is the measure of the chapel, ie. a fourteenth part of the heptagon.

We may conclude by saying that the measure of the radial chapel is implemented as an algorithm, by means of integers, thus determining all the apse measures (100 spans deep and 150 spans wide). The cathedral thus built has a 2/3 ratio, also called fifth, hèmiolion, diapente o sesquilateral ratio, which is fully harmonised with the music of the spheres, where the side chapels and the radial chapels have a 9/8 tonal ratio, in full agreement with the proportions mentioned in Plato's Timaeus.

The proportions of the cathedral's apse are based on a fundamental unit: the measure of the radial chapel. This measure, which is related to the apse width, was well known to the clergymen (through their Neoplatonic sources) and was somehow transmitted to medieval master builders. Therefore, the apse construction is based on applying a proportions theory with a mathematical base. In the cosmological order of Johannes Kepler's Harmonices Mundi, Libri V, Guarç's ratio would be the 9/8 ratio between Jupiter and Mars, or simply the 9/8 tonal proportion of medieval music.

References

1. Bayerri, E. (1962). Los Códices Medievales de la Catedral de Tortosa. Novísimo inventario descriptivo. Tortosa: Talleres Gráficos Algueró y Baiges, pp.473-474.
2. Bechmann, R. (1991). Villard de Honnecourt. La pensée tecnique au XIIIe siècle et sa communication. Paris: Picard Éditeur, pp. 146-148.
3. Dürer, Albrecht (1525). Underweysung der Messung, mit dem Zirckel und Richtscheyt in Linien, Ebenen und gantzen Corporen. Nuremberg: Hieronymum Formschneyder, f.27v-28v.
4. Gauss, C. F. (1801) Disquisitiones Arithmeticae.

Auctore D. Carolo Federico Gauss. Lipsiae: In Commissis apud Gerh. Fleischer Jun, pp.454-463.
5. Kepler, J. (1864). Joannis Kepleri Astronomi Opera Omnia. Volumen Quintum. Edidit Dr. Ch. Frisch. Francofurti A.M. et Erlangae: Heyder & Zimmer, pp. V5:101-108.
6. Ketâb al-nejârat (Sur ce qui est indispensable aux artisans dans les constructions géométriques,) Teheran: Written Heritage Research Centre & Institut Français de Recherche en Iran, proposición W47, (2010).
7. Lluis i Ginovart, Josep; et. alt. (2013). Gothic construction and the traça of a heptagonal apse. The problem of the heptagon. "Nexus Network Journal: Architecture and Mathematics", vol. 15, I. 2, pp 325-348, doi: 10.1007/s00004-013-0152-x.
8. Lluis i Ginovart; Costa (2014), Design and medieval construction: The case of Tortosa cathedral (1345-1441). Construction History Society. Volume 29, No.1, pp. 1-24.
9. Lluis i Ginovart, Josep, (2014). La sciencia theorica speculativa en la construcción gótica. El promotor eclesiástico de la catedral medieval. Hispania Sacra. Hispania Sacra, LXVI, Extra I, enero-junio 2014, pp. 7-34, doi: 10.3989/hs.2014.069.
10. Saidan A. S. (1974). The Arithmetic of Abū'l-Wafā'. "Isis", vol. 65, No. 3, pp. 367-375. Statement (P2, C.IV-V).
11. Sarrade M. T. (1986). Sur les connaissances mathématiques des bâtisseurs de cathédrales. Paris: Librairie du Compagnonnage, pp. 43-55.
12. Woepcke, M. F. (1855). Recherches sur l´historie des Sciences mathématiques chez les orientaux, dápres des traités inedits Arabes et Persans. Deuxième siecle. Analyse et extrait d'un recueil de constructions géométriques par Abûl Wafa. "Journal asiatique", Fevier-Avril. Vol.V, pp. 218-256, pp. 309-359, statements (CII.6) and (CIII.13). Suter, H. (1922). Das Buch der geometrischen Konstruktionen. Suter H. Das Buch der geometrischen Konstruktionen des Abu'l Wefa. "Abhandlungen zur Geschichte der Naturwissenschaften und Medizin", no. 4, pp. 94-109. Aghayani-Chavoshi (2010).