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FUZZY IDEALS AND FUZZY DOT IDEALS ON BH-ALGEBRAS

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ABSTRACT

In this paper we introduce the notions of Fuzzy Ideals in BH-algebras and the notion of fuzzy dot Ideals of BH-algebras and investigate some of their results.

Keywords: BH-algebras, BH- Ideals, Fuzzy Dot BH-ideal.

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1 INTRODUCTION

Y. Imai and K. Iseki [1, 2, and 3] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. K. Iseki and S. Tanaka, [7] are introduced Ideal theory of BCK-algebras P. Bhattacharya, N.P. Mukherjee and L.A. Zadeh [4] are introduced fuzzy relations and fuzzy groups. The notion of BH-algebras is introduced by Y. B Jun, E. H. Roh and H. S. Kim[9] Since then, several authors have studied BH-algebras. In particular, Q. Zhang, E. H. Roh and Y. B. Jun [10] studied the fuzzy theory in BH-algebras. L.A. Zadeh [6] introduced notion of fuzzy sets and A. Rosenfeld [8] introduced the notion of fuzzy group. O.G. Xi [5] introduced the notion of fuzzy BCK-algebras. After that, Y.B. Jun and J. Meng [10] studied Characterization of fuzzy sub algebras by their level sub algebras on BCK-algebras. J. Neggers and H. S. Kim[11] introduced on *d*-algebras, M. Akram [12] introduced on fuzzy *d*-algebras In this paper we classify the notion of Fuzzy Ideals on BH – algebras and the notion of Fuzzy dot

Ideals on BH – algebras. And then we investigate several basic properties which are related to fuzzy BH-ideals and fuzzy dot BH- ideals

2 PRELIMINARIES

In this section we cite the fundamental definitions that will be used in the sequel:

Definition 2.1 [1, 2, 3]

Let X be a nonempty set with a binary operation $*$ and a constant 0. Then (X, $*$, 0) is called a BCK-algebra if it satisfies the following conditions

1. $((x*y)*(x*z))*(z*y)=0$
2. $(x*(x*y))*y=0$
3. $x*x=0$
4. $x*y=0, y*x=0 \Rightarrow x=y$
5. $0*x=0$ $for \ \ all\, x,y,z \in X$

Definition 2.2 [1, 2, 3]

Let X be a BCK-algebra and I be a subset of X, then I is called an ideal of X if

(I1) 0$\in I$

(I2) $y \ and \ x*y \in I \Rightarrow x \in I \ for \ all \ x,y \in I$

Definition 2.3 [9, 10]

A nonempty set X with a constant 0 and a binary operation $*$ is called a BH-algebra, if it satisfies the following axioms

(BH1) $x*x=0$

(BH2) $x*0=0$

(BH3) $x*y=0 \ and \ y*x=0 \Rightarrow x=y$ for all $x,y \in X$

Example 2.4

Let X= {0, 1, 2} be a set with the following cayley table

*	0	1	2
0	0	1	1
1	1	0	1
2	2	1	0

Then (X, $*, 0$) is a BH-algebra

Definition 2.5[9, 10]

Let X be a BH-algebra and I be a subset of X, then I is called an ideal of X if

(BHI1) 0$\in I$

(BHI2) $y \ and \ x*y \in I \Rightarrow x \in I$

(BHI3) $x \in I$ and $y \in X \Rightarrow x*y \in I \ for \ all \ x,y \in I$

A mapping $f: X \rightarrow Y$ of BH-algebras is called a homomorphism if $f\,(x*y\,) = f(x)*f(y)$ for all $x, y\, X \in X$. Note that if $f : X \rightarrow Y$ is homomorphism of BH-algebras, Then $f\,(0) = 0'$. We now review some fuzzy logic concepts. A fuzzy subset of a set X is a function $\mu : X \rightarrow [0,1]$. For a fuzzy subset μ of X and t ∈ [0, 1], define U (μ; t) to be the set U (μ; t) = {x ∈ X | μ(x) ≥ t}. For any fuzzy subsets μ and ν of a set X, we define

$$(\mu \cap \nu)(x) = min\{\mu(x), \nu(x)\} \ for \ all \ x \in X.$$

Let $f : X \rightarrow Y$ be a function from a set X to a set Y and let μ be a fuzzy subset of X. The fuzzy subset ν of Y defined by $\nu(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \mu(x) & if \quad f^{-1}(y) \neq \phi, \quad \forall y \in Y \\ 0 & otherwise \end{cases}$

is called the image of μ under f, denoted by $f(\mu)$. If ν is a fuzzy subset of Y, the fuzzy subset μ of X given by $\mu(x) = \nu(f(x))$ for all x ∈ X is called the Preimage of ν under f and is denoted by $f^{-1}(\nu)$. A fuzzy subset μ in X has the sup property if for any $T \subseteq X$ there exists x_0∈ T such that $\mu(x_0) = \sup_{x \in f^{-1}(y)} \mu(z)$. A fuzzy relation μ on a set X is a fuzzy subset of $X \times X$, that is, a map $\mu : X \times X \rightarrow [0, 1]$.

Definition 2.6[4, 6, 8]

Let X be a nonempty set. A fuzzy (sub) set μ of the set X is a mapping $\mu: X \rightarrow [0,1$

Definition 2.7[4, 6, 8]

Let μ be the fuzzy set of a set X. For a fixed s∈ [0,1], the set $\mu_s = \{x \in X: \mu(x) \geq s\}$is called an upper level of μ or level subset of μ

Definition 2.8[5, 7]

A fuzzy set μ in X is called fuzzy BCK-ideal of X if it satisfies the following inequalities

1.$\mu(0) \geq \mu(x)$

2. $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$

Definition 2.9 [11]

Let X be a nonempty set with a binary operation ∗ and a constant 0. Then (X, ∗, 0) is called a d - algebra if it satisfies the following axioms.

1.$x * x = 0$

2. $0 * x = 0$

3. $x * y = 0, y * x = 0 \Rightarrow x = y$ for all $x, y \in X$

Definition 2.10 [12]

A fuzzy set μ in X is called fuzzy d-ideal of X if it satisfies the following inequalities

Fd1.$\mu(0) \geq \mu(x)$

Fd2$\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$

Fd3. $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$ For all $x, y \in X$

Definition 2.11[12] A fuzzy subset μ of X is called a fuzzy dot d-ideal of X if it satisfies The following conditions:

1. $\mu(0) \geq \mu(x)$
2. $\mu(x) \geq \mu(x * y) \,.\, \mu(y)$
3. $\mu(x * y) \geq \mu(x). \mu(y)$for all $x, y \in X$

3. FUZZY IDEALS ON BH-ALGEBRAS

Definition 3.1

A fuzzy set μ in X is called fuzzy BH-ideal of X if it satisfies the following inequalities

1.$\mu(0) \geq \mu(x)$

2. $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$

3. $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$

Example 3.2

Let X= {0, 1, 2, 3} be a set with the following cayley table

*	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	2	0	0
3	3	3	3	0

Then (X, $*$,0) is not BCK-algebra. Since $\{(1* 3) * (1 * 2)\} * (2 * 3) = 1 \neq 0$

We define fuzzy set μ in X by $\mu(0) = 0.8$ and $\mu(x) = 0.01$ for all $x \neq 0$ in X. then it is easy to show that μ is a BH-ideal of X.

We can easily observe the following propositions

1. In a BH-algebra every fuzzy BH-ideal is a fuzzy BCK-ideal, and every fuzzy BCK-ideal is a fuzzy BH -Sub algebra
2. Every fuzzy BH-ideal of a BH- algebra is a fuzzy BH-subalgebra.

Example 3.3

Let X = {0, 1, 2} be a set given by the following cayley table

*	0	1	2
0	0	0	0
1	1	0	1
2	2	2	0

Then (X, $*$,0) is a fuzzy BCK-algebra. We define fuzzy set μ in X by $\mu(0) = 0.7, \mu(1) = 0.5, \mu(2) = 0.2$

Then μ is a fuzzy BH-ideal of X.

Definition 3.4

Let λ and μ be the fuzzy sets in a set X. The Cartesian product $\lambda \times \mu: X \times X \rightarrow [0,1]$ is defined by $(\lambda \times \mu)(x, y) = \min\{\lambda(x), \mu(y)\}$ $\forall x, y \in X$.

Theorem 3.5

If λ and μ be the fuzzy BH-ideals of a BH- algebra X, then $\lambda \times \mu$ is a fuzzy BH-ideals of $X \times X$

Proof

For any $(x, y) \in X \times X$, wehave

$(\lambda \times \mu)(0,0) = \min\{\lambda(0), \mu(0)\} \geq \min\{\lambda(x), \mu(y)\}$

$= (\lambda \times \mu)(x, y)$

That is $(\lambda \times \mu)(0,0) = (\lambda \times \mu)(x, y)$

Let (x_1, x_2) and $(y_1, y_{2)} \in X \times X$

Then, $(\lambda \times \mu)(x_1, x_2) = min\{\lambda(x_1), \mu(x_2)\}$

$\geq \min\{\min\{\lambda(x_1 * y_1), \lambda(y_1)\}, \min\{\mu(x_2 * y_2), \mu(y_2)\}$

$= \min\{\min \lambda(x_1 * y_1), \mu(x_2 * y_2), \min\{\lambda(y_1), \mu(y_2)\}\}$

$= \min\{(\lambda \times \mu)(x_1 * y_1, x_2 * y_2)), (\lambda \times \mu)(y_1, y_2)\}$

$= \min\{((\lambda \times \mu)(x_1, x_2) * (y_1, y_2)), (\lambda \times \mu)(y_1, y_2)\}$

That is $((\lambda \times \mu)(x_1, x_2)) = \min\{(\lambda \times \mu)(x_1, x_2) * (y_1, y_2)), (\lambda \times \mu)(y_1, y_2)\}$

And $(\lambda \times \mu)((x_1, x_2) * (y_1, y_2))$

= $(\lambda \times \mu)(x_1 * y_1, x_2 * y_2)$
= min{ $\lambda(x_1 * y_1), \mu(x_2 * y_2)$}
≥ min {min {λ (x_1), λ (y_1)}, min {$\mu(x_2), \mu(y_2)$}}
= min {min (λ (x_1), $\mu(x_2)$, min {(λ (y_1), $\mu(y_2)$)}
= min {$(\lambda \times \mu)((x_1, x_2), (\lambda \times \mu)(y_1, y_2))$
That is $(\lambda \times \mu)((x_1, x_2) * (y_1, y_2))$
=min {$(\lambda \times \mu)((x_1, x_2), (\lambda \times \mu)(y_1, y_2)$}
Hence $\lambda \times \mu$ is a fuzzy BH-ideal of X× X

Theorem 3.6

Let λ $and\, \mu$ be fuzzy sets in a BH-algebra such that $\lambda \times \mu$ is a fuzzy BH-ideal of $X \times X$.Then

i)Either λ(0)≥ λ (x)$or\ \mu(0) \geq \mu(x)\ \forall x \in X$.
ii) If λ (0)≥ λ (x) $\forall x \in X$, then either $\mu(0) \geq \lambda(x)$ or $\mu(0) \geq \mu(x)$
iii) If $\mu(0) \geq \mu(x)\ \forall x \in X$, then either λ (0)≥ λ (x) or λ (0)≥ μ(x)

Proof

We use reduction to absurdity
i) Assume λ(x)> λ (0) and $\mu(x) \geq \mu(0)$ for some $x, y \in X$.
Then $(\lambda \times \mu)(x, y)$ = min{λ(x), $\mu(y)$}
>min {{λ(0), $\mu(0)$}
= $(\lambda \times \mu)(0,0)$
$(\lambda \times \mu)(x, y) > (\lambda \times \mu)(0,0)\ \forall x, y \in X$
Which is a contradiction to $(\lambda \times \mu)$ is a fuzzy BH-ideal of X× X
Therefore either λ (0)≥ λ (x)$or\ \mu(0) \geq \mu(x)\ \forall x \in X$.
ii) Assume $\mu(0) < \lambda(x)$ and $\mu(0) < \mu(y)$ for some $x, y \in X$.
Then $(\lambda \times \mu)(0,0)$ = min{λ(0), $\mu(0)$}= $\mu(0)$
And $(\lambda \times \mu)(x, y)$ = min{λ(x), $\mu(y)$}> $\mu(0)$
= $(\lambda \times \mu)(0,0)$
This implies $(\lambda \times \mu)(x, y) >$ and $(\lambda \times \mu)(0,0)$
Which is a contradiction to $\lambda \times \mu$ is a fuzzy BH-ideal of X× X
Hence if λ (0)≥ λ (x) $\forall x \in X, then$
Either $\mu(0) \geq \lambda(x)$ or $\mu(0) \geq \mu(x)\ \ \forall x \in X$
iii) Assume λ (0)< λ (x) or λ (0)< μ(y) $\forall x, y \in X$
Then (($\lambda \times \mu$)(0, 0) = min{λ(0), $\mu(0)$}= $\lambda(0)$
And $(\lambda \times \mu)(x, y)$ = min{λ(x), $\mu(y)$}> $\lambda(0)$
= $(\lambda \times \mu)(0,0)$
This implies $(\lambda \times \mu)(x, y) > (\lambda \times \mu)(0,0)$
Which is a contradiction to $(\lambda \times \mu)$ is a fuzzy BH-ideal of $X \times X$
Hence if $\mu(0) \geq \mu(x)\ \ \forall x \in X$ then either λ (0)≥ λ (x) or λ (0)≥ μ(x)
This completes the proof

Theorem 3.7

If $\lambda \times \mu$ is a fuzzy BH-idela of $X \times X$ then λ or μ is a fuzzy BH-ideal of X.

Proof

First we prove that μ is a fuzzy BH-ideal of X.

Given $\lambda \times \mu$ is a fuzzy BH-ideal of $X \times X$, then by theorem 3.6(i), either $\lambda(0) \geq \lambda(x)$ *or* $\mu(0) \geq \mu(x)\ \forall x \in X$.

Let $\mu(0) \geq \mu(x)$

By theorem 3.6(iii) then either $\lambda(0) \geq \lambda(x)$ or $\lambda(0) \geq \mu(x)$

Now $\mu(x) = \min\{\lambda(0), \mu(x)\}$

$= (\lambda \times \mu)(0, x)$

$\geq \min\{ ((\lambda \times \mu)(0,x) * (0,y)),\ (\lambda \times \mu)(0,y)\}$

$= \min\{ (\lambda \times \mu)(0*0), x*y), (\lambda \times \mu)(0,y)\}$

$= \min\{ (\lambda \times \mu)(0, x*y), (\lambda \times \mu)(0,y)\}$

$= \min\{ (\lambda \times \mu)(0*0), x*y), (\lambda \times \mu)(0,y)\}$

$= \min\{ \mu(x*y), (\mu)(y)\}$

That is $\mu(x) \geq \min\{ \mu(x*y), \mu(y)\}$

$\mu(x*y) = \min\{ \lambda(0), \mu(x*y)\}$

$= (\lambda \times \mu)(0, x*y)$

$= (\lambda \times \mu)(0*0, x*y)$

$= (\lambda \times \mu)(0,x) * (0,y)$

$\mu(x*y) \geq \min\{(\lambda \times \mu)(0,x), (\lambda \times \mu)(0,y)\}$

$= \min\{\mu(x), \mu(y)\}$

That is, $\mu(x*y) \geq \min\{\mu(x), \mu(y)\}$

This proves that μ is a fuzzy BH-ideal of X.

Secondly to prove that λ is a Fuzzy BH-ideal of X.

Using theorem 4.6(i) and (ii) we get

This completes the proof.

Theorem 3.8

If μ is a fuzzy BH-idela of X, then μ_t is a BH-idela of X for all $t \in [0,1]$

Proof

Let μ be a fuzzy BH-ideal of X,

Then By the definition of BH-ideal

$$\mu(0) \geq \mu(x)$$

$\mu(x) \geq \min\{ \mu(x*y), (\mu(y)\}$

$\mu(x*y) \geq \min\{ \mu(x), (\mu(y)\}\ \ \forall x, y \in X$

To prove that μ_t *is a* BH-ideal of x.

By the definition of level subset of μ

$\mu_t = \{x / \mu(x) \geq t\}$

Let $x, y \in \mu_t$ and μ is a fuzzy BH-ideal of X.

Since $\mu(0) \geq \mu(x) \geq t$ implies $0 \in \mu_t$, for all $t \in [0,1]$

Let $x, y \in \mu_t$ and $y \in \mu_t$

Therefore $\mu(x*y) \geq t$ and $\mu(y) \geq t$

Now $\mu(x) \geq \min\{ \mu(x*y), (\mu(y)\}$

$\geq$ min $\{t, t\} \geq t$

Hence $(\mu(x) \geq t$

That is $x \in \mu_t$.

Let $x \in \mu_t, y \in X$

Choose y in X such that $\mu(y) \geq t$

Since $x \in \mu_t$ implies $\mu(x) \geq t$

We know that $\mu(x * y) \geq$ min $\{ \mu(x), (\mu(y)\}$

$\geq \min\{t, t\}$

$\geq t$

That is $\mu(x * y) \geq t$ implies $x * y \in \mu_t$

Hence μ_t is a BH-ideal of X.

Theorem 3.9

If X be a BH-algebra, $\forall t \in [0,1]$ and μ_t $is\ a\ BH$ $-$ideal of X, then μ is a fuzzy BH-ideal of X.

Proof

Since μ_t $is\ a\ BH$ $-$ideal of X

0$\in \mu_t$

ii) $x * y \in \mu_t$ And $y \in \mu_t$ implies $x \in \mu_t$

iii)$x \in \mu_t$, y $\in X$ implies $x * y \in \mu_t$

To prove that $\mu\ is\ a\ fuzzy$ BH-ideal of X.

Let $x, y \in \mu_t$ then $\mu(x) \geq t$ and $\mu(y) \geq t$

Let $\mu(x) = t_1$ and $\mu(y) = t_2$

Without loss of generality let $t_1 \leq t_2$

Then $x \in \mu_{t_1}$

Now $x \in \mu_{t_1}$ and y $\in X\ implies\ x * y \in \mu_{t_1}$

That is $\mu(x * y) \geq t_1$

= min $\{t_1, t_2\}$

= min $\{ \mu(x), \mu(y)\}$

$\mu(x * y) \geq$ min $\{ \mu(x), \mu(y)\}$

$ii)\ Let$ $\mu(0) = \mu(x * x) \geq$ min $\{ \mu(x), \mu(y)\} \geq \mu(x)$ (by proof (i))

That is $\mu(0) \geq \mu(x)$ for all x $\in X$

iii) Let $\mu(x) = \mu(x * y) * (0 * y))$

$\geq \min\{ \mu(x * y), \mu(0 * y)\big)$ (By (i))

$\geq \min\{ \mu(x * y), \min\{\mu(0), \mu(y)\}$

$\geq \min\{ \mu(x * y), \{\mu(y)\big)$ (By (ii))

$\mu(x) \geq \min\{ \mu(x * y), \{\mu(y)\big)$

Hence μ is a fuzzy BH-ideal of X.

Definition 3.10

A fuzzy set μ in X is said to be fuzzy BH- $\boldsymbol{\chi}$ ideal if $\mu(x * u * v * y) \geq \min\{ \mu(x), \mu(y)\}$

Theorem 3.11

Every Fuzzy BH -ideal is a fuzzy BH- **χ**- ideal

Proof

It is trivial

Remark

Converse of the above theorem is not true. That is every fuzzy BH-**χ** -ideal is not true. That is every fuzzy BH -ideal. Let us prove this by an example

Let X = {0, 1, 2} be a set given by the following cayley table

*	0	1	2
0	0	1	2
1	1	0	1
2	2	2	0

Then (X,$*$,0) is a BH-algebra. We define fuzzy set: X→ [0,1] by $\mu(0) = 0.8, \mu(x) = 0.2\ \forall x \neq 0$ clearly μ is a fuzzy BH-ideal of X But μ is not a BH-**χ**-ideal of X.

For Let $x = 0\ \ u = 1\ \ v = 1\ y = 1$

$\mu(x * u * v * y = \mu(0 * 1 * 1 * 1) = \mu(1) = 0.2$

$\min\{\ \mu(x),\ \mu(y)\} = \min\{\ \mu(0),\ \mu(0)\}$

$= \ \mu(0) = 0.8$

$\mu(x * u * v * y)\ \leq \min\ \{\ \mu(x), \mu(y)\}$

Hence μ is not a fuzzy BH-**χ**-ideal of X.

And $\mu(x * y) = \mu\left(f(a) * f(b)\right)$

$\geq\ \mu\ (f(a * b))$

$=\ \mu^f(a * b)$

$\geq \min\{\ \mu^f(a),\ \mu^f(b)\}$

$= \min\{\mu(f(a), \mu(f(b))\}$

$= \min\{\mu(x), \mu(y)\}$

Hence $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$

Hence μ is a fuzzy BH- **χ** ideal of Y.

4. FUZZY DOT *BH*-IDEALS OF *BH*-ALGEBRAS

Definition 4.1. A fuzzy subset μ of *X* is called a fuzzy dot *BH*-ideal of *X* if it satisfies

The following conditions:

(FBH1). $\mu(0) \geq \mu(x)$

(FBH2). $\mu(x) \geq \mu(x * y)\ .\ \mu(y)$

(FBH3). $\mu(x * y) \geq \mu(x).\ \mu(y)$for all $x, y \in X$

Example 4.2. Let *X* = {0, 1, *2*, *3*} be a *BH*-algebra with Cayley table (Table 1) as follows:

(Table 1)

*	0	1	2	3
0	0	0	0	0
1	1	0	0	2
2	2	2	0	0
3	3	3	3	0

Define $\mu : X \to [0,1]$ by μ (0) =0.9, μ (*a*) = μ (*b)* =0.6, μ (*c)* =0.3 .It is easy to verify that μ is a Fuzzy dot *BH*-ideal of *X*.

Proposition 4.3. Every fuzzy *BH*-ideal is a fuzzy dot *BH*-ideal of a *BH*-algebra.

Remark. The converse of Proposition 4.3 is not true as shown in the following Example 4.2**.** Let $X = \{0 ,1 ,2 ,3\}$ be a *BH*-algebra with Cayley table (Table 1) as follows:

Example 4.4. Let $X = \{0,1, 2\}$ be a *BH*-algebra with Cayley table (Table 2) as follows:

(Table 2)

*	0	1	2
0	0	0	0
1	1	0	2
2	2	1	0

Define μ: X → [0, 1] by μ (0) =0.8, μ (1) = 0.5, μ (2) =0.4. It is easy to verify that μ is a fuzzy Dot *BH*-ideal of *X*, but not a fuzzy *d*-ideal of *X* because

$\mu(x) \leq \min\{\mu(x * y), \mu(y)\}$

$\mu(1) = \min\{\mu(1 * 2), \mu(2)\}$

$= \mu(2)$

Proposition 4.5. Every fuzzy dot *BH*-ideal of a *BH*-algebra *X* is a fuzzy dot subalgebra of *X*.

Remark. The converse of Proposition 4.5 is not true as shown in the following Example:

Example 4.6. Let *X* be the BH-algebra in Example 4.4 and define μ: X → [0, 1] by μ (0)= μ (1)=0.9, μ (2)= 0.7. It is easy to verify that μ is a fuzzy dot sub algebra of *X*, but not a fuzzy dot *BH*-ideal of *X* because μ (2)=0.7 ≤ 0.81=μ (2 ∗1)· μ (1) .

Proposition 4.7. If μ and ν are fuzzy dot *BH*-ideals of a *BH*-algebra *X*, then so is μ ∩ ν.

Proof. Let *x, y* ∈ *X*. Then

$$\begin{aligned}(\mu \cap \nu)(0) &= min\,\{\mu\,(0), \nu\,(0)\}\\ &\geq min\,\{\mu(x), \nu(x)\}\\ &= (\mu \cap \nu)\,(x).\end{aligned}$$

Also,
$$\begin{aligned}(\mu \cap \nu)\,(x) &= min\,\{\mu(x), \nu(x)\}\\ &\geq min\,\{\mu(x * y) \cdot \mu(y), \nu(x * y) \cdot \nu(y)\}\\ &\geq (min\,\{\mu(x * y), \nu(x * y)\}) \cdot (min\,\{\mu(y), \nu(y)\})\\ &= ((\mu \cap \nu)\,(x * y)) \cdot ((\mu \cap \nu)\,(y)).\end{aligned}$$

$$\begin{aligned}And, (\mu \cap \nu)\,(x * y) &= min\,\{\mu(x * y), \nu(x * y)\}\\ &\geq min\,\{\mu(x) \cdot \mu(y), \nu(x) \cdot \nu(y)\}\\ &\geq (min\,\{\mu(x), \nu(x)\}) \cdot (min\,\{\mu(y), \nu(y)\})\\ &= ((\mu \cap \nu)\,(x)) \cdot ((\mu \cap \nu)\,(y)).\end{aligned}$$

Hence μ ∩ ν is a fuzzy dot *BH*-ideal of a *d*-algebra *X*.

Theorem 4.8. If each nonempty level subset *U* (*μ; t*) of *μ* is a fuzzy *BH*-ideal of *X* then μ is a fuzzy dot *BH*-ideal of *X*, where t ∈[0, 1].

Definition 4.9

Let σ be a fuzzy subset of X. The strongest fuzzy σ-relation on BH-algebra X is the fuzzy subset μ_σ of X ×X given by μ_σ(x, y) = σ(x) · σ(y) for all x, y ∈ X. A fuzzy relation μ on BH-algebra X is called a Fuzzy σ-product relation if μ(x, y) ≥ σ(x) · σ(y) for all x, y ∈ X. A fuzzy relation μ on BH-algebra is called a left fuzzy relation on σ if μ(x, y) = σ(x) for all x, y ∈ X.

Note that a left fuzzy relation on σ is a fuzzy σ-product relation.

Remark. The converse of Theorem 4.8 is not true as shown in the following example:

Example 4.10. Let *X* be the *BH*-algebra in Example 4.4 and define μ: X → [0, 1] by μ (0)=0.6, μ (1)=0.7, μ (2)= 0.8. We know that μ is a fuzzy dot *BH*-ideal of *X*, but *U* (μ; 0.8) = {*x* ∈*X* | μ(*x*) ≥ 0.8} = {2, 2} is not *BH*-ideal of *X* since 0∉*U* (μ; 0.8).

Theorem 4.11. Let $f : X \to X'$ be an onto homomorphism of *BH*-algebras, ν be a fuzzy Dot BH-ideal of *Y*. Then the Preimage $f^{-1}(v)$ of ν under *f* is a fuzzy dot *BH*-ideal of *X*.

Proof. Let *x*∈ *X*,

$f^{-1}(v)(0) = v(f(0)) = v(0')$

$\geq v(f(x)) = f^{-1}(v)(x)$

For any *x*, *y*∈ *X*, we have

$f^{-1}(v)(x)$ = ν (f (x)) ≥ ν (*f* (*x*)* *f*(*y*)) · ν (*f* (*y*))

= ν (*f* (*x** *y*)) · ν (*f* (*y*)) = $f^{-1}(v)(x * y).f^{-1}(v)(y)$

Also,

$f^{-1}(v)(x * y)$= ν (f (*x* * *y*)) = ν (*f* (*x*) * *f* (*y*))

≥ ν (*f* (*x*)) · ν(*f* (*y*)) = $f^{-1}(v)(x).f^{-1}(v)(y)$

Thus $f^{-1}(v)$is a fuzzy dot *BH*-ideal of *X*.

Theorem: 4.12 An onto homomorphic image of a fuzzy dot *BH*-ideal with the sup Property is a fuzzy dot *BH*-ideal.

Theorem 4.13. If λ and μ are fuzzy dot *BH*-ideal of a *BH*-algebra *X*, then λ × μ is a fuzzy Dot *BH*-ideal of *X* × *X*.

Proof.

Let $x, y \in X$

λ × μ (0,0)=λ(0), μ(0)

≥ λ(x).μ(y)= (λ × μ) (x, y)

For any $x, x', y, y' \in X$ wehave

$(\lambda \times \mu)(x, y) = \lambda(x).\mu(y)$

$\geq \lambda(x * x').\lambda(x'))\mu(y * y').\mu(y')$

$= \lambda(x * x').\mu(y * y')).\lambda(x').\mu(y')$

$= (\lambda\times \mu)(x, y) * (x', y')).(\lambda\times \mu)(x', y')$

Also $(\lambda\times \mu)(x, y) * (x', y')) = (\lambda\times \mu)(x * x') * (y * y'))$.

$= \lambda(x * x').\mu(y * y')$

$\geq \lambda(x).\lambda(x')).(\mu(y).\mu(y'))$

$(\lambda(x).\mu(y)).(\lambda(x').\mu(y'))$

$= (\lambda\times \mu)(x, y).(\lambda \times \mu)(x', y')$

Hence λ × μ is a fuzzy dot *BH*-ideal of *X* × *X*.

Theorem 4.14. Let σ be a fuzzy subset of a *BH*-algebra *X* and $\mu_\sigma$be the strongest fuzzy σ -relation on *BH*-algebra X. Then σ is a fuzzy dot *BH*-ideal of *X* if and only if μ_σ is a Fuzzy dot *BH*-ideal of *X* × *X*.

Proof. Assume that σ is a fuzzy dot BH-ideal of X. For any x, y ∈ X we have

μ_σ (0, 0)= σ (0) · σ (0) ≥ σ(*x*) · σ(*y*) = μ_σ (*x*, *y*).

Let *x*, *x* ′, *y*, *y* ′∈ *X*. Then

$$\begin{aligned} &\mu_\sigma((x,x') * (y,y')) \cdot \mu_\sigma(y,y') \\ &= \mu_\sigma(x * y, x' * y') \cdot \mu_\sigma(y,y') \\ &= (\sigma(x * y) \cdot \sigma(x' * y')) \cdot (\sigma(y) \cdot \sigma(y')) \\ &= (\sigma(x * y) \cdot \sigma(y)) \cdot (\sigma(x' * y') \cdot \sigma(y')) \\ &\le \sigma(x) \cdot \sigma(x') = \mu_\sigma(x,x'). And, \end{aligned}$$

$$\begin{aligned} \mu_\sigma(x, x') \cdot \mu_\sigma(y,y') &= (\sigma(x) \cdot \sigma(x')) \cdot (\sigma(y) \cdot \sigma(y')) \\ &= (\sigma(x) \cdot \sigma(y)) \cdot (\sigma(x') \cdot \sigma(y')) \\ &\le \sigma(x * y) \cdot \sigma(x' * y') \\ &= \mu_\sigma(x * y, x' * y') = \mu_\sigma((x,x') * (y,y')). \end{aligned}$$

Thus μ_σ is a fuzzy dot *BH*-ideal of *X* × *X*.

Conversely suppose that μ_σ is a fuzzy dot *BH*-ideal of *X* × *X*. From (FBH1) we get

$(\sigma(0))^2 = \sigma(0) \cdot \sigma(0) = c\,(0, 0)$

And so σ (0) ≥ σ(*x*) for all *x* ∈ *X*. Also we have

$$\begin{aligned} (\sigma(x))^2 &= \mu_\sigma((x, x) \\ &\ge \mu_\sigma((x, x)*(y, y)).\, \mu_\sigma(y, y) \\ &= \mu_\sigma((x*y), (x * y)).\, \mu_\sigma(y, y) \\ &= ((\sigma(x * y) \cdot \sigma(y)))^2 \end{aligned}$$

Which implies that $\sigma(x) \ge \sigma(x * y) \cdot \sigma(y)$ for all *x*, *y* ∈ *X*.

Also we have

$$\begin{aligned} (\sigma(x * y)^2 &= \mu_\sigma((x*y), x * y) \\ &= \mu_\sigma\big((x,x) * (y,y)\big) \ge \mu_\sigma(x,x).\, \mu_\sigma(y,y) \\ &= (\sigma(x) \cdot \sigma(y))^2 \end{aligned}$$

So $\sigma(x * y) \ge \sigma(x) \cdot \sigma(y)$ for all *x*, *y* ∈*X*.

Therefore σ is a fuzzy dot *BH*-ideal of *X*.

Proposition 4.15. Let μ be a left fuzzy relation on a fuzzy subset σ of a *BH*-algebra *X*. If μ is a fuzzy dot BH-ideal of *X* × *X*, then σ is a fuzzy dot BH-ideal of a BH-algebra X.

Proof.

Suppose that a left fuzzy relation μ on σ is a fuzzy dot BH-ideal of *X* × *X*.

Then

σ (0) = μ (0, *z*) ∀z ∈ *X*

By putting *z*=0

σ (0) = μ (0,0) ≥ μ (*x* , *y*)=σ (*x*),

For all *x*∈ *X*.

For any *x*, *x* ′, *y*, *y* ′∈ *X*

$\sigma(x) = \mu(x, y) \geq \mu((x, y) * (x', y')).\mu(x', y')$

$= \mu((x * x'), (y * y')).\mu(y, y'))$

$= \sigma(x * x').\sigma(x')$

Also

$\sigma(x * x').= \mu(x * x', y * y') = \mu((x, y) * (x', y'))$

$\geq \mu(x, y).\mu(x', y')$

$=\sigma(x).\sigma(x')$

Thus σ is a fuzzy dot BH-ideal of a BH-algebra X.

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