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UNIFORM ATTRACTORS FOR VANISHING VISCOSITY APPROXIMATIONS OF NON-AUTONOMOUS COMPLEX FLOWS

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Abstract. In this paper we prove the existence of uniform global attractors in the strong topology of the phase space for semiflows generated by vanishing viscosity approximations of some class of non-autonomous complex fluids.

Key words: non-Newtonian fluids, parabolic equations, global attractors, infinite-dimensional dynamical systems..

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1. Introduction

In this paper we consider a non-autonomous evolution problem which appears in the investigation of the model of concentrated suspensions (proposed by Hebraud and Lequex [12]) with non-autonomous coefficients. More precisely, the unknown function $p(x,t)$, representing probability density, satisfies the following equation:

$$\frac{\partial p}{\partial t} = -b(t)\frac{\partial p}{\partial x} + D(p)\frac{\partial^2 p}{\partial x^2} - \chi_{\mathbb{R}\setminus[-1,1]}(x)p + \frac{D(p)}{\alpha}\delta_0(x), \tag{1.1}$$

where $\alpha > 0$ is a parameter, $\chi_{\mathbb{R}\setminus[-1,1]}$ is the characteristic function of the open set $\mathbb{R}\setminus[-1,1]$, δ_0 is the Dirac delta function with support at the origin,

$$D(f) = \alpha \int\limits_{|x|>1} f(x)dx,$$

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and the function $b(t)$ is assumed to be non-autonomous. Moreover, mechanical background of the model requires boundedness with respect to the time of the average stress function

$$\tau(t) = \int_{\mathbb{R}} xp(t,x)dx.$$

Existence and uniqueness results for such model were proved in [4]. The theory of global attractors was applied first for (1.1) in Amigó et al. [1], where the existence of global unbounded attractors with respect to the weak topology was proved for the case $b(t) \equiv 0$. Numerical aspects were investigated in [2,13]. The key point in [4,13] was the analysis of the so-called vanishing viscosity approximation system, where the diffusion coefficient was everywhere positive. In [3, 5–10, 14–22] the existence of global attractor in the strong topology of the phase space for m-semiflow generated by vanishing viscosity approximation was proved. Only autonomous (i.e. $b(t) \equiv const$) case was considered. In the present paper we extend results from [14] to much more general non-autonomous case, using the uniform global attractor approach [11, 23–26].

2. Setting of the problem and preliminaries

Let $\alpha > 0$ be a positive constant, $0 \le \varepsilon \ll 1$ be a small parameter, and $b : \mathbb{R}_+ \to \mathbb{R}$ be a measurable function. Consider the following evolution problem with non-degenerate diffusion:

$$\frac{\partial p}{\partial t} = -b(t)\frac{\partial p}{\partial x} + (D(p)+\varepsilon)\frac{\partial^2 p}{\partial x^2} - \chi_{\mathbb{R}\setminus[-1,1]}(x)p + \frac{D(p)}{\alpha}\delta_0(x), \text{ a.e. in } \mathbb{R}\times\mathbb{R}_+; \tag{2.1}$$

$$p(x,t) \ge 0, \text{ a.e. in } \mathbb{R}\times\mathbb{R}_+; \tag{2.2}$$

$$\int_{\mathbb{R}} p(x,t)dx = 1, \text{ a.e. in } \mathbb{R}_+; \tag{2.3}$$

$$\int_{\mathbb{R}} |x|p(x,t)dx < \infty, \text{ a.e. in } \mathbb{R}. \tag{2.4}$$

Suppose that b is an essentially bounded function, that is, there exists a constant $B > 0$ such that

$$|b(t)| \le B \text{ for a.e. } t > 0. \tag{2.5}$$

Further we will use the following notation:

$$L^p = L^p(\mathbb{R}), \;\; H^1 = H^1(\mathbb{R}), \;\; H^{-1} = (H^1)^*,$$

for each $1 \le p \le \infty$. Let $\langle \cdot\,,\cdot \rangle$ be the pairing on $H^{-1}\times H^1$ (on $L^q \times L^p$ respectively with $p \ge 1$ and $1 < q \le \infty$ such that $\frac{1}{p}+\frac{1}{q}=1$) that coincides with the inner product on L^2, that is,

$$\langle f, u\rangle = \int_{\mathbb{R}} f(x)u(x)dx,$$

for each $f \in L^2$ and $u \in H^1$ (for each $f \in L^q$ and $u \in L^p$, respectively).

Let $0 \le \tau < T < \infty$ be arbitrary fixed. A solution of equation (2.1) on a finite time interval $[\tau, T]$ is defined as follows.

Definition 2.1. Let $0 < \varepsilon \ll 1$. A function $p \in L^\infty(\tau, T; L^1 \cap L^2) \cap L^2(\tau, T; H^1)$ with $\frac{\partial p}{\partial t} \in L^2(\tau, T; H^{-1})$ is called a (weak) solution of equation (2.1) on $[\tau, T]$, if the equality

$$\int_\tau^T \left(\langle \frac{\partial p}{\partial t}, \eta \rangle + b(t) \langle \frac{\partial p}{\partial x}, \eta \rangle + (D(p(\,\cdot\,, t)) + \varepsilon) \langle \frac{\partial p}{\partial x}, \frac{\partial \eta}{\partial x} \rangle + \int_{|x|>1} p \cdot \eta \, dx \right) dt$$
$$= \int_\tau^T \frac{D(p(\,\cdot\,, t))}{\alpha} \langle \delta_0, \eta \rangle dt, \tag{2.6}$$

holds for each $\eta \in L^2(\tau, T; H^1)$.

Remark 2.1. We note that the right hand-side of equality (2.6) is equal to

$$\int_\tau^T \frac{D(p(t))}{\alpha} \eta(0, t) dt.$$

Remark 2.2. Let $0 < \varepsilon \ll 1$, and p be a solution of equation (2.1) on $[\tau, T]$. Since $p \in L^2(\tau, T; H^1)$ and $\frac{\partial p}{\partial t} \in L^2(\tau, T; H^{-1})$, then $p \in C([\tau, T]; L^2)$, and, therefore, the following initial condition

$$p|_{t=\tau} = p_\tau(x), \text{ a.e. in } \mathbb{R}, \tag{2.7}$$

makes sense for $p_\tau \in L^1 \cap L^2$.

Let

$$X := \{p \in L^2(\mathbb{R}) \, : \, \int_\mathbb{R} |x| \, |p(x)| \, dx < \infty\},$$

which is a Banach space with the norm

$$\|p\|_X := \|p\|_{L^2} + \int_\mathbb{R} |x| \, |p(x)| \, dx, \quad p \in X.$$

Remark 2.3. The embedding $X \subset L^1 \cap L^2$ is continuous. Moreover, $X = \overline{L}^1 \cap L^2$, where

$$\overline{L}^1 := \{p \in L^1 \, : \, \int_\mathbb{R} |x| \, |p| \, dx < \infty\}$$

is a Banach space with the following norm:

$$\|p\|_{\overline{L}^1} := \int_\mathbb{R} (1 + |x|) \, |p| \, dx, \quad p \in \overline{L}^1.$$

We understand condition (2.4) in the sense of the following definition.

Definition 2.2. The solution p of equation (2.1) on $[\tau, T]$ satisfies condition (2.4) on $[\tau, T]$ if $xp \in L^\infty(\tau, T; L^1)$.

Remark 2.4. Let p be a solution of equation (2.1) on $[\tau, T]$. Then $xp \in L^\infty(\tau, T; L^1)$ if and only if $p \in L^\infty(\tau, T; X)$. Moreover, since $p \in L^\infty(0, T; X)$, $p \in C([0, T]; L^2)$, and $X \subset L^2$, we have that $p \in C([0, T]; X_w)$.

Let $0 < \varepsilon \ll 1$ be arbitrary fixed. Cancès et al. [4, Proposition 2.1] proved that for each p_τ such that

$$p_\tau \in L^1 \cap L^\infty, \;\; p_\tau \geq 0, \;\; \int_{\mathbb{R}} p_\tau(x)dx = 1, \;\; \int_{\mathbb{R}} |x| p_\tau(x)dx < \infty, \tag{2.8}$$

problem (2.1)–(2.4), (2.7) on $[\tau, T]$ has a unique solution p. Moreover,

$$\begin{gathered} p \in L^\infty(\mathbb{R} \times (\tau, T)), \;\; \sigma p \in L^\infty\left(0, T; L^1\right), \\ p \in C([\tau, T]; L^2 \cap L^1), \;\; D(p) \in C([\tau, T]), \end{gathered}$$

and

$$\int_{\mathbb{R}} p\left(t, \sigma\right) d\sigma = 1, \;\; p\left(t\right) \geq 0 \text{ for all } t \geq 0. \tag{2.9}$$

Therefore, the phase space for this problem can be defined as follows:

$$H := \mathrm{cl}_X E, \;\; E := \{p \in X \,:\, p \in L^\infty, \, p \geq 0, \, \int_{\mathbb{R}} p(x)dx = 1\},$$

where cl_X is the closure in the space X (see Amigó et al. [1]). The convexity of E implies the equality $H = \mathrm{cl}_{X_w} E$.

Remark 2.5. For $0 < \varepsilon \ll 1$ it is easy to show that for every $p_\tau \in E$ $p \in C([\tau, T]; (L^1 \cap L^\infty)_w)$. In particular, we have that $p(t) \in E$ for each $t \in [\tau, T]$. Therefore, for each $p \in H$ the following two conditions hold: (a) $p(x) \geq 0$ for a.e. $x \in \mathbb{R}$, and (b) $\int_{\mathbb{R}} p(x)dx = 1$ [1, p. 212]. Moreover, for each $0 < \varepsilon \ll 1$, $0 \leq \tau < T < \infty$, and $p_\tau \in H$ there exists no more than one solution p of problem (2.1)-(2.3), (2.7) on $[\tau, T]$.

The main goal of the present paper is to show the existence of uniform global attractors in the strong topology of the phase space H for the m-semiflow generated by the non-autonomous problem (2.1)–(2.4).

3. Existence and properties of solutions

In this section we provide results from [14] about existence and topological properties of (2.1)-(2.4).

Let $\mathcal{K}^+_{\tau,\varepsilon}$ ($\mathcal{D}^+_{\tau,\varepsilon}$) denotes the family of all globally defined solutions of problem (2.1)-(2.3) ((2.1)-(2.4)) on $[\tau, \infty)$ with $p\left(\tau\right) \in H$. By definition, $\mathcal{D}^+_{\tau,\varepsilon} \subseteq \mathcal{K}^+_{\tau,\varepsilon}$

Lemma 3.1. [14, Lemma 3.1] *There exists a constant $C > 0$ such that, if*

$$0 \le \varepsilon \ll 1,\ \tau \ge 0 \text{ and } p \in \mathcal{K}^+_{\tau,\varepsilon} \text{ with } p(\tau) \in H,$$

then $p \in \mathcal{D}^+_{\tau,\varepsilon}$ and the following inequality holds:

$$\|p(t)\|_{\overline{L}^1} \le \|p(\tau)\|_{\overline{L}^1} e^{-\frac{1}{2}(t-\tau)} + C, \tag{3.1}$$

for each $t \ge \tau$. Moreover, for each $\delta > 0$ and a bounded set (in $\overline{L}^1$) $K \subset H$ there exist constants $T = T(\delta, K) > 0$ and $\bar{k} = \bar{k}(\delta, K) > 0$ such that for each $0 \le \varepsilon \ll 1$, $\tau \ge 0$, and $p \in \mathcal{K}^+_{\tau,\varepsilon}$ with $p(\tau) \in K$ the following inequality holds:

$$\int_{|x|>2k} p(x,t)|x|dx \le \delta, \tag{3.2}$$

for each $t \ge \tau + T$ and $k \ge \bar{k}$.

Remark 3.1. According to Lemma 3.1, each globally defined solution p of problem (2.1)–(2.3) on $[\tau, \infty)$ with $\tau \ge 0$, $0 \le \varepsilon \ll 1$, and $p(\tau) \in H$, belongs to $L^\infty(\tau, \infty; \overline{L}^1)$. In particular, the following equality holds:

$$\mathcal{D}^+_{\tau,\varepsilon} = \{p \in \mathcal{K}^+_{\tau,\varepsilon} \,:\, p(\tau) \in H\}.$$

The following result guaranties existence and dissipativity for the problem (2.1)-(2.4).

Theorem 3.1. *Let $0 < \varepsilon \ll 1$. Then for every $p_\tau \in H$ problem (2.1)–(2.4), (2.7) on $[\tau, T]$ has a unique solution p. Moreover, $p \in C([\tau, T]; H)$. Moreover, there exists $R_0 > 0$ such that for an arbitrary bounded (in L^2) set $K \subset H$ and for arbitrary $\varepsilon \in (0, 1)$ there exists a moment of time $T = T(K, \varepsilon)$ such that for every $\tau \ge 0$ and $p \in \mathcal{D}^+_{\tau,\varepsilon}$ satisfying $p(\tau) \in K$ the following inequality holds:*

$$\|p(t)\|_{L^2} \le R_0, \tag{3.3}$$

for each $t \ge \tau + T$.

The next result guaranties the continuous properties of solutions of (2.1)-(2.4).

Theorem 3.2. [14, Lemma 3.3] *Let $0 \le \tau < T < \infty$, $p^n_\tau \in H$, $b_n \in L^\infty(\tau, T)$, and $0 < \varepsilon_n \ll 1$ for each $n = 0, 1, \ldots$. Suppose that $|b_n(t)| \le B$ for a.e. $t \in (\tau, T)$ and $p^n \in C([\tau, T]; H_w)$ be a solution of problem (2.1)–(2.4), (2.7) on $[\tau, T]$ with parameters $p^n_\tau, \varepsilon_n, b_n$, for each $n \ge 1$. If*

$$p^n_\tau \to p^0_\tau \text{ in } H_w,\ \varepsilon_n \to \varepsilon_0 > 0,\ b_n \to b_0 \text{ weakly-star in } L^\infty(\tau, T),$$

then there exists a solution $p \in C([\tau, T]; H_w)$ of problem (2.1)–(2.4), (2.7) on $[\tau, T]$ with parameters $p^0_\tau, \varepsilon_0, b_0$, such that up to a subsequence the following convergence holds:

$$p^n \to p \text{ in } C([\tau, T]; H_w). \tag{3.4}$$

Moreover, if $p^n_\tau \to p^0_\tau$ in H, then the following statements hold:

(a) $p, p^n \in C([\tau, T]; H)$ *for each* $n \geq 1$;

(b) the following convergence holds for the entire sequence:

$$p^n \to p \text{ in } L^2(\tau, T; H^1), \tag{3.5}$$

$$p^n \to p \text{ in } C([\tau, T]; H). \tag{3.6}$$

If, additionally, $b_n \to b_0$ *in the Lebesgue measure on* $[\tau, T]$, *then*

$$\frac{\partial p^n}{\partial t} \to \frac{\partial p}{\partial t} \text{ in } L^2(\tau, T; H^{-1}). \tag{3.7}$$

4. Existence and properties of uniform global attractors in the non-autonomous case

To characterize the uniform long-time behavior of solutions for non-autonomous dissipative dynamical system consider the *united trajectory space* $\mathcal{K}^+_{\varepsilon,\cup}$ for the family of solutions $\{\mathcal{K}^+_{\varepsilon,\tau}\}_{\tau\geq 0}$ shifted to zero:

$$\mathcal{K}^+_{\varepsilon,\cup} := \bigcup_{\tau\geq 0} \left\{T(h)y(\,\cdot + \tau\,) \,:\, y(\,\cdot\,) \in \mathcal{K}^+_{\varepsilon,\tau},\, h \geq 0\right\}, \tag{4.1}$$

and the *extended united trajectory space* for the family $\{\mathcal{K}^+_{\varepsilon,\tau}\}_{\tau\geq 0}$:

$$\mathcal{K}^+_{\varepsilon} := \mathrm{cl}_{C^{\mathrm{loc}}(\mathbb{R}_+;H)}\left[\mathcal{K}^+_{\varepsilon,\cup}\right], \tag{4.2}$$

where $\mathrm{cl}_{C^{\mathrm{loc}}(\mathbb{R}_+;H)}[\,\cdot\,]$ is the closure in $C^{\mathrm{loc}}(\mathbb{R}_+; H)$. Since $T(h)\mathcal{K}^+_{\varepsilon,\cup} \subseteq \mathcal{K}^+_{\varepsilon,\cup}$ for each $h \geq 0$, then

$$T(h)\mathcal{K}^+_{\varepsilon} \subseteq \mathcal{K}^+_{\varepsilon} \text{ for each } h \geq 0, \tag{4.3}$$

due to

$$\rho_{C^{\mathrm{loc}}(\mathbb{R}_+;H)}(T(h)u, T(h)v) \leq \rho_{C^{\mathrm{loc}}(\mathbb{R}_+;H)}(u, v) \text{ for each } u, v \in C^{\mathrm{loc}}(\mathbb{R}_+; H),$$

where $\rho_{C^{\mathrm{loc}}(\mathbb{R}_+;H)}$ is the standard metric on Fréchet space $C^{\mathrm{loc}}(\mathbb{R}_+; H)$. Therefore the set

$$\mathbb{X} := \{y(0) \,:\, y \in \mathcal{K}^+_{\varepsilon}\} \tag{4.4}$$

is closed in H. We endow this set $\mathbb{X}$ with metric

$$\rho_{\mathbb{X}}(x_1, x_2) = \|x_1 - x_2\|_X, \quad x_1, x_2 \in \mathbb{X}.$$

Then we obtain that $(\mathbb{X},\rho)$ is a Polish space (complete separable metric space).

Let us define the multivalued semiflow (*m-semiflow*) $V_\varepsilon : \mathbb{R}_+ \times \mathbb{X} \to 2^{\mathbb{X}}$:

$$V_\varepsilon(t, y_0) := \{y(t) \,:\, y(\cdot) \in \mathcal{K}^+_{\varepsilon} \text{ and } y(0) = y_0\}, \quad t \geq 0,\, y_0 \in \mathbb{X}. \tag{4.5}$$

According to (4.3) and (4.4) for each $t \geq 0$ and $y_0 \in \mathbb{X}$ the set $V_\varepsilon(t, y_0)$ is nonempty. Moreover, the following two conditions hold:

(i) $V_\varepsilon(0,\cdot) = I$ is the identity map;

(ii) $V_\varepsilon(t_1+t_2, y_0) \subseteq V_\varepsilon(t_1, V_\varepsilon(t_2, y_0))$, $\forall t_1, t_2 \in \mathbb{R}_+$, $\forall y_0 \in \mathbb{X}$,

where $V_\varepsilon(t, D) = \underset{y\in D}{\cup} V_\varepsilon(t,y)$, $D \subseteq \mathbb{X}$.

We denote by $\mathrm{dist}_{\mathbb{X}}(C, D) = \sup_{c\in C}\inf_{d\in D}\rho_{\mathbb{X}}(c,d)$ the *Hausdorff semidistance* between nonempty subsets C and D of the Polish space $\mathbb{X}$. Recall that the compact set $\Theta_\varepsilon \subset \mathbb{X}$ is a *global attractor* of the m-semiflow V_ε if it satisfies the following conditions:

(i) Θ_ε attracts each bounded subset $B \subset \mathbb{X}$, i.e.

$$\mathrm{dist}_{\mathbb{X}}(V_\varepsilon(t,B), \Theta_\varepsilon) \to 0, \quad t \to +\infty; \tag{4.6}$$

(ii) Θ_ε is negatively semi-invariant set, that is, $\Theta_\varepsilon \subseteq V_\varepsilon(t, \Theta_\varepsilon)$ for each $t \geq 0$.

In this paper we examine the uniform long-time behavior of solution sets $\{\mathcal{K}^+_{\tau,\varepsilon}\}_{\tau\geq 0}$ in the strong topology of the natural phase space H (as time $t \to +\infty$ for a fixed $\varepsilon > 0$) in the sense of the existence of a compact global attractor for m-semiflow V_ε generated by the family of solution sets $\{\mathcal{K}^+_{\tau,\varepsilon}\}_{\tau\geq 0}$ and their shifts.

Theorem 4.1. *For each $\varepsilon > 0$ the m-semiflow (4.5) has the connected stable global attractor Θ_ε in the phase space $\mathbb{X}$. Moreover, Θ_ε is bounded in H uniformly in ε.*

Proof. Due to Theorems 3.1, 3.2 and classical results about existence of global attractors (see [21]) it is sufficient to prove that V_ε is asymptotically compact, that is,

$$\text{every sequence } \{\bar{\xi}_n \in V_\varepsilon(t_n, p_0^n)\} \text{ is precompact in } H,$$

where $t_n \nearrow +\infty$, $\|p_0^n\|_X \leq r$.

Let $\bar{\xi}_n \in V_\varepsilon(t_n, p_0^n)$. Then $\exists \xi_n : \|\xi_n - \bar{\xi}_n\|_{\mathbb{X}} < \frac{1}{n}$ and $\xi_n = p_n(t_n)$, p_n is a solution of (2.1)–(2.4) with $p_n(0) = p_0^n$ and $b_n(\cdot) := b(\cdot + \tau_n)$, $\tau_n \geq 0$. Therefore, from Theorem 3.1

$$\|p_n(t)\|_X \leq R_0 + r, \ \forall\, n \geq 1, \ t \geq 0. \tag{4.7}$$

So we can claim that $\{\xi_n\}$ is precompact in H_w. Indeed, since $\|\xi_n\|_{L^2} \leq R_0 + r$ then up to subsequence $\xi_n \to \xi$ in L^2_w. Let us prove that up to a subsequence $\xi_n \to \xi$ in $\overline{L}^1_w$. Since $\xi_n = p_n(t_n)$, then (3.2) yields that for each $\delta > 0$ there exist $k(\delta) \geq 1$, $n(\delta) \geq 1$ such that

$$\int_{|x|>k} \xi_n(x)|x|dx < \frac{\delta}{3}, \ \forall\, k \geq k(\delta), \ n \geq n(\delta).$$

According to Amigó et al. [1, Lemma 6.1]

$$(\overline{L^1})^* = \{\varphi = (1+|x|)u \, : \, u \in L^\infty\}.$$

Thus, we set $d_n(x) = (1+|x|)\xi_n(x)$ and prove that $\{d_n\}$ is a Cauchy sequence in L^1_w, because

$$\left|\int_{\mathbb{R}}(d_n(x)-d_m(x))u(x)dx\right| \le \left|\int_{|x|\le k}(1+|x|)(\xi_n(x)-\xi_m(x))u(x)dx\right|$$
$$+\,2\|u\|_{L^\infty}\left(\int_{|x|>k}\xi_n(x)|x|dx+\int_{|x|>k}\xi_m(x)|x|dx\right)<\delta,$$

for each $u \in L^\infty$ and $n, m \ge N = N(\delta, k)$. Since the space L^1 is weakly complete, then up to a subsequence $d_n \to d$ in L^1_w for some $d \in L^1$. Thus

$$\xi_n \to \bar{\xi} = \frac{d}{1+|x|} \ \text{ in } \ \overline{L}^1_w.$$

If we consider the restriction of ξ_n to each interval $[-k, k]$, then we deduce that $\bar{\xi} = \xi$ and up to a subsequence $\xi_n \to \xi$ in H_w.

Now let us prove this convergence in the strong topology of H. Consider a smooth real function θ that satisfies the following three conditions:

$$\begin{array}{lll} \text{(a)} & \theta(s)=0, & |s|\le 1; \\ \text{(b)} & 0\le\theta(s)\le 1, & |s|\in[1,2]; \\ \text{(c)} & \theta(s)=1, & |s|\ge 2, \end{array} \tag{4.8}$$

and define for $k > 1$

$$\rho_k(x) = \theta(\frac{x}{k}).$$

According to Amigó et al. [1, pp. 215–216] after multiplying (2.1) by $\rho_k(x)p_n$ we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\rho_k(x)p_n^2dx + b_n(t)\int_{\mathbb{R}}\rho_k(x)p_n\frac{\partial p_n}{\partial x}dx \\ &+(D(p_n(\cdot,t))+\varepsilon_n)\Big(\int_{\mathbb{R}}\rho_k(x)\left(\frac{\partial p_n}{\partial x}\right)^2dx \\ &+\frac{1}{k}\int_{\mathbb{R}}\theta'(\frac{x}{k})p_n\frac{\partial p_n}{\partial x}dx\Big)+\int_{\mathbb{R}}\rho_k(x)p_n^2dx=0. \end{aligned} \tag{4.9}$$

Integrating by parts we deduce

$$b_n(t)\int_{\mathbb{R}}(\rho_k(x)p_n\frac{\partial p_n}{\partial x}dx = -\frac{b_n(t)}{2k}\int_{\mathbb{R}}\theta'(\frac{x}{k})p_n^2dx,$$

$$\frac{1}{k}\int_{\mathbb{R}}\theta'(\frac{x}{k})p_n\frac{\partial p_n}{\partial x}dx = -\frac{1}{2k^2}\int_{\mathbb{R}}\theta''(\frac{x}{k})p_n^2dx.$$

Then from (4.9) we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\rho_k(x)p_n^2dx+\int_{\mathbb{R}}\rho_k(x)p_n^2dx \le \left(\frac{B\beta}{2k}+\frac{(\alpha+1)\beta}{2k^2}\right)\int_{\mathbb{R}}p_n^2dx, \tag{4.10}$$

where $\beta := \max_{|s|\in[1,2]} \{|\theta'(s)| + |\theta''(s)|\}$.

Combining (4.7) and (4.10) we deduce from Gronwall's Lemma that for some positive constant $C = C(r)$

$$\int_{|x|>2k} p_n^2(x,t)dx \le e^{-2t} r^2 + \frac{C(r)}{k},\ \forall\ t \ge 0,\ \ n \ge 1,\ \ k > 1. \tag{4.11}$$

On the other hand, for every solution of (2.1)–(2.4) we have the following energy equality (for details see the proof of Lemma 3.2):

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} (p(x,t))^2 dx + (D(p(\,\cdot\,,t)) + \varepsilon)\int_{\mathbb{R}} \left(\frac{\partial p(x,t)}{\partial x}\right)^2 dx + \int_{|x|>1} (p(x,t))^2 dx$$
$$= \frac{D(p(\,\cdot\,,t))}{\alpha}\langle \delta_0, p(\,\cdot\,,t)\rangle. \tag{4.12}$$

Let us consider the functions

$$\bar{p}_n(t) = p_n(t + (t_n - 1)),\ \ t \ge 0.$$

Then $\bar{p}_n$ is a solution of (2.1)–(2.4) with $\bar{b}_n(\,\cdot\,) := b_n(\,\cdot\, + t_n - 1) = b(\,\cdot\, + t_n - 1 + \tau_n)$, $\bar{p}_n(0) = p_n(t_n - 1)$, $\bar{p}_n(1) = \xi_n$ and $\bar{p}_n$ satisfies (4.7), (4.9), (4.12). Moreover, similarly to the previous arguments we deduce that up to subsequence

$$\bar{p}_n(0) = p_n(t_n - 1) \to \bar{p}_0 \ \text{ in } \ H_w.$$

Hence, from Lemma 3.2 we obtain for every $T > 1$ that

$$\bar{p}_n \to \bar{p} \ \text{ in } \ C([0,T]; H_w), \tag{4.13}$$

where $\bar{p}$ is a solution of (2.1)–(2.4) with $\bar{p}(0) = \bar{p}_0$ and some $\bar{b} \in L^\infty(0, +\infty)$ such that $\bar{b}_n \to b$ weakly star in $L^\infty(0,T)$ for each $T > 0$. In particular, $|\bar{b}(t)| \le B$ for a.e. $t > 0$.

Since $\varepsilon > 0$ is fixed, we can derive from (4.7), (4.12) and the Aubin-Lions theorem [16] that for every $k > 1$ up to subsequence

$$\bar{p}_n \to \bar{p} \ \text{ in } \ L^2(0,T; L^2(-k,k)).$$

In particular,

$$\bar{p}_n(t) \to \bar{p}(t) \ \text{ in } \ L^2(-k,k) \ \text{ for a.a. } t \in (0,T).$$

By a diagonal procedure we obtain that up to a subsequence and for some $\tau \in (0,1)$,

$$\bar{p}_n(\tau) \to \bar{p}(\tau) \ \text{ in } \ L^2(-k,k),\ \forall\ k \ge 1. \tag{4.14}$$

From (4.11) we get

$$\int_{|x|>2k} \bar{p}_n^2(x,\tau)dx \le e^{-2(\tau + t_n - 1)} r^2 + \frac{C(r)}{k},\ \forall\ n \ge 1,\ \ k > 1. \tag{4.15}$$

Combining (3.2), (4.14), (4.15) we have

$$\bar{p}_n(\tau) \to \bar{p}(\tau) \quad \text{in} \quad X.$$

Then the second part of Theorem 3.2 guarantees the convergence

$$\bar{p}_n \to \bar{p} \quad \text{in} \quad C([\tau, T]; H).$$

In particular,

$$\xi_n = \bar{p}_n(1) \to \bar{p}(1) \quad \text{in } H.$$

Thus we obtain the required precompactness of $\{\xi_n\}$ and, therefore, the existence of the connected, stable global attractor Θ_ε. □

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