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Fractional Quadruple Laplace Transform and its Properties

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ABSTRACT
In this paper, we introduce definition for fractional quadruple Laplace transform of order $\alpha, 0 < \alpha \le 1$, for fractional differentiable functions. Some main properties and inversion theorem of fractional quadruple Laplace transform are established. Further, the connection between fractional quadruple Laplace transform and fractional Sumudu transform are presented.

KEYWORD: *quadruple Laplace transform, Sumudu transform, fractional Difference.*

INTRODUCTION
There are different integral transforms in mathematics which are used in astronomy, physics and also in engineering. The integral transforms were vastly applied to obtain the solution of differential equations; therefore there are different kinds of integral transforms like Mellin, Laplace, and Fourier and so on. Partial differential equations are considered one of the most significant topics in mathematics and others. There are no general methods for solve these equations. However, integral transform method is one of the most familiar methods in order to get the solution of partial differential equations [1, 2]. In [3, 9] quadruple Laplace transform and Sumudu transforms were used to solve wave and Poisson equations. Moreover the relation between them and their applications to differential equations have been determined and studied by [5, 6]. In this study we focus on quadruple integral determined and studied by [5, 6]. In this study we focus on quadruple integral transforms. First of all, we start to recall the definition of quadruple Laplace transform as follows.

Definition: Let f be a continuous function of four variables; then the quadruple Laplace transform of $f(w,x,y,z)$ is defined by

$$L_{wxyz} f(w,x,y,z) = F(p,q,r,s) \qquad (1)$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{-pw} e^{-qx} e^{-ry} e^{-sz} f(w,x,y,z) dw dx dy dz$$

Where $w,x,y,z > 0$ and p,q,r,s are Laplace variables, and

$$f(w,x,y,z) = \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} e^{pw} \left[\frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} e^{qx} \left[\frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{ry} \left[\frac{1}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} e^{sz} F(p,q,r,s) dz\right] dy\right] dx\right] dw$$

is the inverse Laplace transform.

Fractional Derivative via Fractional Difference
Definition: let $g(t)$ be a continuous function, but not necessarily differentiable function, then the forward operator $FW(\mathrm{h})$ is defined as follows $FW(\mathrm{h})g(t) = g(t+h)$, Where $h > 0$ denote a constant discretization span.

Moreover, the fractional difference of $g(t)$ is known as

$$\Delta^\alpha g(t) = (FW - h)^\alpha g(t)$$
$$= \sum_{m=0}^{\infty} (-1)^m \binom{\alpha}{m} g(t + (\alpha - m)h)$$

Where $0 < \alpha < 1$,
And the α-derivative of $g(t)$ is known as

$$g^{(\alpha)}(t) = \lim_{h \downarrow 0} \frac{\Delta^{\alpha} g(t)}{h^{\alpha}}$$

See the details in [9, 10].

Modified Fractional Riemann-Liouville Derivative

The author in [10] proposed an alternative definition of the Riemann-Liouville derivative

Definition: let $g(t)$ be a continuous function, but not necessarily differentiable function, then

Let us presume than $g(t) = K$, where K is a constant, thus α-derivative of the function $g(t)$ is $D_t^{\alpha} K = \begin{cases} K\Gamma^{-1}(1-\alpha)t^{-\alpha} & , \alpha \le 0, \\ 0, & otherwise. \end{cases}$

On the other hand, when $g(t) \neq K$ hence $g(t) = g(0) + \big(g(t) - g(0)\big)$,

and fractional derivative of the function $g(t)$ will become known as

$$g^{(\alpha)}(t) = D_t^{\alpha} g(0) + D_t^{\alpha}\big(g(t) - g(0)\big),$$

at any negative α, (α < 0) one has

$$D_t^{\alpha}\big(g(t) - g(0)\big) = \frac{1}{\Gamma(-\alpha)} \int_0^t (t-\eta)^{-\alpha-1} g(\eta) d\eta, \alpha < 0,$$

While for positive α, we will put

$$D_t^{\alpha}\big(g(t) - g(0)\big) = D_t^{\alpha} g(t) = D_t\big(g^{(\alpha-1)}\big).$$

When $m < \alpha < m+1$, we place

$$g^{(\alpha)}(t) = (g^{(\alpha-m)}(t))^{(m)}, \quad m \le \alpha < m+1, m \ge 1.$$

Integral with Respect to $(dt)^{\alpha}$

The next lemma show the solution of fractional differential equation

$$dy = g(t)(dt)^{\alpha}, t \ge 0, y(0) = 0 \tag{2}$$

By integration with respect to $(dt)^{\alpha}$

Lemma: If $g(t)$ is a continuous function, so the solution of (2) is defined as the following

$$y(t) = \int_0^1 g(\eta)(d\eta)^{\alpha}, \ \ y(0) = 0$$

$$= \alpha \int_0^1 (t-\eta)^{\alpha-1} g(\eta) d\eta, \ \ 0 < \alpha < 1 \tag{3}$$

For more results and varies views on fractional calculus, see for example [15, 16, 17, 18, 19, 20, 21]

Fractional quadruple Sumudu Transform

Recently, in [13] the author defined quadruple sumudu transform of the function depended on two variables. Analogously, fractional quadruple sumudu transform was defined and some properties were given as the following

Definition: [14] The fractional quadruple sumudu transform of function $f(x,t)$ is known as

$$S_{\alpha}^4\{ f(x,y,z,t)\} = G_{\alpha}^4(p,q,r,s),$$

$$= \int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} E_{\alpha}(-(x+y+z+t)^{\alpha}) f(px,qy,rz,st)(dx)^{\alpha}(dy)^{\alpha}(dz)^{\alpha}(dt)^{\alpha}$$

Where $p,q,r,s \in \mathbb{C}$, $x,y,z,t > 0$ and

$$E_{\alpha}(x) = \sum_{m=0}^{\infty} \frac{x^m}{\Gamma(\alpha m + 1)}$$

is the Mittag-Leffler function.

Some Properties of Fractional Quadruple Sumudu Transform

We recall some properties of Fractional Quadruple Sumudu Transform

$$S_{\alpha}^4\{ f(ax,by,cz,dt)\} = G_{\alpha}^4(ap,bq,cr,ds),$$

$$S_{\alpha}^4\{ f(x-a,y-b,z-c,t-d)\} = E_{\alpha}(-(a+b+c+d)^{\alpha}) G_{\alpha}^4(ap,bq,cr,ds),$$

$$S_{\alpha}^4\{\partial_t^{\alpha} f(ax,by,cz,dt)\} = \frac{G_{\alpha}^4(p,q,r,s) - \Gamma(1+\alpha) f(0,x,y,z)}{u^{\alpha}} \tag{4}$$

where ∂_t^{α} is denoted to fractional partial derivative of order α (see [10]).

Main Results

The main results in this work are present in the following sections

Quadruple Laplace Transform of Fractional order

Definition 6: If $f(x,y,z,t)$ is a function where $x,y,z,t > 0$, then Quadruple Laplace Transform of Fractional order of $f(x,y,z,t)$ is defined as

$$L_{\alpha}^4\{f(x,y,z,t)\} = F_{\alpha}^4(p,q,r,s) \tag{5}$$

$$= \int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} E_{\alpha}(-(px+qy+rz+st)^{\alpha}) f(x,y,z,t)(dx)^{\alpha}(dy)^{\alpha}(dz)^{\alpha}(dt)^{\alpha}$$

Where $p,q,r,s \in \mathbb{C}$ and $E_{\alpha}(x)$ is the Mittag-Leffler function.

Corollary 10.1 By using the Mittag-Leffler property then we can rewrite the formula (5) as the following:

$$L_\alpha^4\{f(x,y,z,t)\} = F_\alpha^4(p,q,r,s)$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty E_\alpha(-(px)^\alpha) E_\alpha(-qy^\alpha) E_\alpha(-(rz)^\alpha) E_\alpha(-(st)^\alpha) f(x,y,z,t)(dx)^\alpha (dy)^\alpha (dz)^\alpha (dt)^\alpha \quad (6)$$

Remark 8: In particular case, fractional quadruple Laplace transform (5) turns to quadruple Laplace transform (1) when α = 1.

Some properties of fractional quadruple Laplace transform

In this section, various properties of fractional quadruple Laplace transform are discussed and proved such as linearity property, change of scale property and so on.

1. Linearity property

Let $f_1(x,y,z,t)$ and $f_2(x,y,z,t)$ be functions of the variables x and t, then

$$L_\alpha^4\{a_1 f_1(x,y,z,t) + a_2 f_2(x,y,z,t)\}$$
$$= a_1\ L_\alpha^4\{f_1(x,y,z,t)\} + a_2\ L_\alpha^4\{f_2(x,y,z,t)\} (7)$$

Where a_1 and a_2 are constants.

Proof: We can simply get the proof by applying the definition.

2. Changing of scale property

If $L_\alpha^4\{f(x,y,z,t)\} = F_\alpha^4(p,q,r,s)$ hence

$$L_\alpha^4\{f(ax,by,cz,dt)\} = \frac{1}{a^\alpha b^\alpha c^\alpha d^\alpha} F_\alpha^4\left(\frac{p}{a},\frac{q}{b},\frac{r}{c},\frac{s}{d}\right)$$

Whenever a and b are constants.

Proof:

$$\{f(ax,by,cz,dt)\}$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty E_\alpha(-(px+qy+rz+st)^\alpha) f(ax,by,cz,dt)\,(dx)^\alpha (dy)^\alpha (dz)^\alpha (dt)^\alpha (8)$$

We set $j = ax$, $k = by, l = cz$ and $m = dt$ in the above equality (8), therefore we obtain

$$L_\alpha^4\{f(ax,by,cz,dt)\}$$

$$= \frac{1}{a^\alpha b^\alpha c^\alpha d^\alpha} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty E_\alpha\left(-\left(\frac{pj}{a}+\frac{qk}{b}+\frac{rl}{c}+\frac{sm}{d}\right)^\alpha\right) f(j,k,l,m)(dx)^\alpha (dy)^\alpha (dz)^\alpha (dt)^\alpha$$

$$= \frac{1}{a^\alpha b^\alpha c^\alpha d^\alpha} F_\alpha^4\left(\frac{p}{a},\frac{q}{b},\frac{r}{c},\frac{s}{d}\right)$$

3. Shifting property

Let $L_\alpha^4\{f(x,y,z,t)\} = F_\alpha^4(p,q,r,s)$ then

$$L_\alpha^4\{E_\alpha(-(px+qy+rz+st)^\alpha) f(x,y,z,t)\} = F_\alpha^4(p+a, q+b, r+c, s+d)$$

Proof:

$$L_\alpha^4\{E_\alpha(-(ax+by+cz+dt)^\alpha) f(x,y,z,t)\}$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty E_\alpha(-(ax+by+cz+dt)^\alpha) E_\alpha(-(px+qy+rz+st)^\alpha)\ f(x,y,z,t)(dx)^\alpha (dy)^\alpha (dz)^\alpha (dt)^\alpha$$

By using the equality

$$\{E_\alpha(\lambda(x+y+z+t)^\alpha)$$
$$= E_\alpha(\lambda x^\alpha)\ E_\alpha(\lambda y^\alpha) E_\alpha(\lambda z^\alpha) E_\alpha(\lambda t^\alpha)$$

Which implies that

$$L_\alpha^4\{E_\alpha(-(ax+by+cz+dt)^\alpha) f(x,y,z,t)$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty E_\alpha(-((a+p)x+(b+q)y+(c+r)(d+s)t)^\alpha)\ f(x,y,z,t)(dx)^\alpha (dy)^\alpha (dz)^\alpha (dt)^\alpha$$

Hence

$$L_\alpha^4\{E_\alpha(-(px+qy+rz+st)^\alpha) f(x,y,z,t)\}$$
$$= F_\alpha^4(p+a, q+b, r+c, s+d)$$

4. Multiplication by $x^\alpha t^\alpha$

$$L_\alpha^4\{f(x,y,z,t)\} = F_\alpha^4(p,q,r,s)$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty E_\alpha(-(px)^\alpha) E_\alpha(-(qy)^\alpha) E_\alpha(-(rz)^\alpha) E_\alpha(-(st)^\alpha) f(x,y,z,t)(dx)^\alpha (dy)^\alpha (dz)^\alpha (dt)^\alpha$$

then

$$\{x^\alpha y^\alpha z^\alpha t^\alpha f(x,y,z,t)\} = \frac{\partial^{2\alpha}}{\partial p^\alpha \partial q^\alpha \partial r^\alpha \partial s^\alpha} L_\alpha^4\{f(x,y,z,t)\}$$

Proof:

$$L_\alpha^4\{x^\alpha y^\alpha z^\alpha t^\alpha f(x,y,z,t)\}$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty x^\alpha E_\alpha(-(px)^\alpha) E_\alpha(-(qy)^\alpha) E_\alpha(-(rz)^\alpha) E_\alpha(-(st)^\alpha) f(x,y,z,t)(dx)^\alpha (dy)^\alpha (dz)^\alpha (dt)^\alpha$$

By using the fact $D_s^\alpha(E_\alpha(-s^\alpha x^\alpha))$ = -$x^\alpha E_\alpha(-s^\alpha x^\alpha)$, then

$$L_\alpha^4\{x^\alpha y^\alpha z^\alpha t^\alpha f(x,y,z,t)\}$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \frac{\partial^\alpha}{\partial p^\alpha} E_\alpha(-(px)^\alpha) \frac{\partial^\alpha}{\partial q^\alpha} E_\alpha(-(qy)^\alpha) \frac{\partial^\alpha}{\partial r^\alpha} E_\alpha(-(rz)^\alpha \frac{\partial^\alpha}{\partial s^\alpha} E_\alpha(-(st)^\alpha) f(x,y,z,t)(dx)^\alpha (dy)^\alpha (dz)^\alpha (dt)^\alpha$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \frac{\partial^{4\alpha}}{\partial p^\alpha \partial q^\alpha \partial r^\alpha \partial s^\alpha} E_\alpha(-(px)^\alpha) E_\alpha(-(qy)^\alpha) E_\alpha(-(rz)^\alpha) E_\alpha(-(st)^\alpha) f(x,y,z,t)(dx)^\alpha (dy)^\alpha (dz)^\alpha (dt)^\alpha$$

$$= \frac{\partial^{2\alpha}}{\partial p^\alpha \partial q^\alpha \partial r^\alpha \partial s^\alpha} L_\alpha^4\{f(x,y,z,t)\}$$

Remark: All results above are suitable for quadruple Laplace transform when α = 1.

Theorem: If the fractional quadruple Laplace transform of the function $f(x,y,z,t)$ is $L_\alpha^4\{f(x,y,z,t)\} = F_\alpha^4(p,q,r,s)$, and fractional quadruple sumudu transform of the function $f(x,y,z,t)$ is $S_\alpha^4\{ f(x,y,z,t)\} = G_\alpha^4(p,q,r,s)$,then

$$G_\alpha^4(p,q,r,s) = \frac{1}{p^\alpha q^\alpha r^\alpha s^\alpha} F_\alpha^4\left(\frac{1}{p},\frac{1}{q},\frac{1}{r},\frac{1}{s}\right)$$

Proof:

$$G_\alpha^4(p,q,r,s) = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty E_\alpha(-x^\alpha) E_\alpha(-y^\alpha) E_\alpha(-z^\alpha) E_\alpha(-t^\alpha) f(px,qy,rz,st)(dx)^\alpha (dy)^\alpha (dz)^\alpha (dt)^\alpha$$

By using change of variables $j \to px, k \to qy, l \to rz$ and $m \to d$

$$= \frac{1}{p^\alpha q^\alpha r^\alpha s^\alpha} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty E_\alpha\left(-\left(\frac{j}{p}\right)^\alpha\right) E_\alpha\left(-\left(\frac{k}{q}\right)^\alpha\right) E_\alpha\left(-\left(\frac{l}{r}\right)^\alpha\right) E_\alpha\left(-\left(\frac{m}{s}\right)^\alpha\right) f(j,k,l,m)(dj)^\alpha (dk)^\alpha (dl)^\alpha (dm)^\alpha$$

$$G_\alpha^4(p,q,r,s) = \frac{1}{p^\alpha q^\alpha r^\alpha s^\alpha} F_\alpha^4\left(\frac{1}{p},\frac{1}{q},\frac{1}{r},\frac{1}{s}\right)$$

The Convolution Theorem of the Fractional Double Laplace Transform

Theorem: The double convolution of order α of functions $f(x,y,z,t)$ and $g(x,y,z,t)$ can be defined as the expression

$$(f(x,y,z,t) **{**}_\alpha\, g(x,y,z,t)) = \int_0^x \int_0^y \int_0^z \int_0^t f(x-\eta, y-\theta, z-\gamma, t-\delta) g(\eta,\theta,\gamma,\delta)\, (d\eta)^\alpha (d\theta)^\alpha (d\gamma)^\alpha (d\delta)^\alpha,$$

therefore one has the equality

$$L_\alpha^4\{(f **{**}_\alpha\, g)(x,y,z,t)\} = L_\alpha^4\{f(x,y,z,t)\} L_\alpha^4\{f(x,y,z,t)\}$$

Proof: We obtain the definition of fractional quadruple Laplace transform and fractional quadruple convolution above, then we obtain

$$L_\alpha^4\{(f **{**}_\alpha\, g)(x,y,z,t)\} = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty E_\alpha(-(px)^\alpha) E_\alpha(-(qy)^\alpha) E_\alpha(-(rz)^\alpha) E_\alpha(-(st)^\alpha) f(f **{**}_\alpha\, g)(dx)^\alpha (dy)^\alpha (dz)^\alpha (dt)^\alpha$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty E_\alpha(-(px)^\alpha) E_\alpha(-(qy)^\alpha) E_\alpha(-(rz)^\alpha)(dt)^\alpha E_\alpha(-(st)^\alpha) \left[\int_0^x \int_0^y \int_0^z \int_0^t f(x-\eta, y-\theta, z-\gamma, t-\delta) g(\eta,\theta,\gamma,\delta)(d\eta)^\alpha (d\theta)^\alpha (d\gamma)^\alpha (d\delta)^\alpha\right] (dx)^\alpha (dy)^\alpha (dz)^\alpha (dt)^\alpha$$

let $j = x-\eta, k = y-\theta, l = z-\gamma, m = t-\delta$ and taking the limit from 0 to ∞, it gives

$$= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty E_\alpha(-p^\alpha (j+\eta)^\alpha) E_\alpha(-q^\alpha (k+\theta)^\alpha) E_\alpha(-r^\alpha (l+\gamma)^\alpha) E_\alpha(-s^\alpha (m+\delta)^\alpha) \int_0^x \int_0^y \int_0^z \int_0^t f(j,k,l,m) g(\eta,\theta,\gamma,\delta)(d\eta)^\alpha (d\theta)^\alpha (d\gamma)^\alpha (d\delta)^\alpha (dj)^\alpha (dk)^\alpha (dl)^\alpha (dm)^\alpha$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty E_\alpha(-p^\alpha j^\alpha) E_\alpha(-q^\alpha k^\alpha) E_\alpha(-r^\alpha l^\alpha) E_\alpha(-s^\alpha m^\alpha) f(j,k,l,m)(dj)^\alpha (dk)^\alpha (dl)^\alpha (dm)^\alpha \int_0^x \int_0^y \int_0^z \int_0^t E_\alpha(-p^\alpha \eta^\alpha) E_\alpha(-q^\alpha \theta^\alpha) E_\alpha(-r^\alpha \gamma^\alpha) E_\alpha(-s^\alpha \delta^\alpha) g(\eta,\theta,\gamma,\delta)(d\eta)^\alpha (d\theta)^\alpha (d\gamma)^\alpha (d\delta)^\alpha$$

$$L_\alpha^4\{f(x,y,z,t)\} L_\alpha^4\{f(x,y,z,t)\}$$

Inversion formula of Quadruple Fractional Laplace's transform

Firstly, we will set up definition of fractional delta function of two variables as follows

Definition: Two variables delta function $\delta_\alpha(x-a, y-b, z-c, t-d)$ of fractional order α, 0 < α ≤1, can be defined as next formula

$$\int_R \int_R \int_R \int_R g(x,y,z,t)\, \delta_\alpha(x-a, y-b, z-c, t-d)(dx)^\alpha (dy)^\alpha (dz)^\alpha (dt)^\alpha = \alpha^4 g(a,b,c,d) (9)$$

In special case, we have

$$\int_R\int_R\int_R\int_R g(x,y,z,t)\,\delta_\alpha(x,y,z,t)(dx)^\alpha(dy)^\alpha(dz)^\alpha(dt)^\alpha = \alpha^4 g(0,0,0,0)$$

Example: we can obtain fractional quadruple Laplace transform of function $\delta_\alpha(x-a,y-b,z-c,t-d)$ as follows

$$L_\alpha^4\{\delta_\alpha(x-a,y-b,z-c,t-d)\}$$

$$=\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty E_\alpha(-(px+qy+rz+st)^\alpha)\,\delta_\alpha(x-a,y-b,z-c,t-d)(dx)^\alpha(dy)^\alpha(dz)^\alpha(dt)^\alpha$$

$$=\alpha^4 E_\alpha(-(pa+qy+rz+st)^\alpha)(10)$$

In particular, we have $L_\alpha^4\{\delta_\alpha(x,y,z,t)\}=\alpha^2$

Relationship between Two Variables Delta Function of Order α and Mittag-Leffler Function

The relationship between $E_\alpha(x+y+z+t)^\alpha$ and $\delta_\alpha(x,y,z,t)$ is clarified by the following theorem

Theorem: The following formula holds

$$\frac{\alpha^4}{(M_\alpha)^{4\alpha}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(i(-hx)^\alpha E_\alpha(i(-uy)^\alpha E_\alpha(i(-vz)^\alpha E_\alpha(i(-wt)^\alpha(dh)^\alpha(du)^\alpha(dv)^\alpha(dw)^\alpha = \delta_\alpha(x,y,z,t)(11)$$

Where M_α satisfy the equivalence $E_\alpha(i((M_\alpha)^\alpha)$ =1, and it is called period ofthe Mittag-Leffler function.

Proof: We test that (11) agreement with

$$\alpha^2=\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(i(hx)^\alpha E_\alpha(i(uy)^\alpha E_\alpha(i(vz)^\alpha E_\alpha(i(wt)^\alpha(dx)^\alpha(dy)^\alpha(dz)^\alpha(dt)^\alpha$$

We replace $\delta_\alpha(x,y,z,t)$in above equality by (11) to get

$$\alpha^2=\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}(dx)^\alpha(dy)^\alpha(dz)^\alpha(dt)^\alpha\frac{\alpha^4}{(M_\alpha)^{4\alpha}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(i(hx)^\alpha)E_\alpha(i(uy)^\alpha)E_\alpha(i(vz)^\alpha)$$
$$E_\alpha(i(wt)^\alpha)E_\alpha(i(-px)^\alpha E_\alpha(i(-qy)^\alpha E_\alpha(i(-rz)^\alpha E_\alpha(i(-st)^\alpha(dp)^\alpha(dq)^\alpha(dr)^\alpha(ds)^\alpha$$

$$=\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}(dx)^\alpha(dy)^\alpha(dz)^\alpha(dt)^\alpha\frac{\alpha^4}{(M_\alpha)^{4\alpha}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(ix^\alpha(h-p)^\alpha)E_\alpha(iy^\alpha(u-q)^\alpha)$$
$$E_\alpha(iz^\alpha(v-r)^\alpha E_\alpha(it^\alpha(w-s)^\alpha(dp)^\alpha(dq)^\alpha(dr)^\alpha(ds)^\alpha$$

$$=\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}(dx)^\alpha(dy)^\alpha(dz)^\alpha(dt)^\alpha\frac{\alpha^4}{(M_\alpha)^{4\alpha}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(i(-xj)^\alpha)E_\alpha(i(-yk)^\alpha)$$
$$E_\alpha(i(-zl)^\alpha)E_\alpha(i(-tm)^\alpha)(dj)^\alpha(dk)^\alpha(dl)^\alpha(dm)^\alpha$$
$$=\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\delta_\alpha(x,y,z,t)(dx)^\alpha(dy)^\alpha(dz)^\alpha(dt)^\alpha$$
$$=\alpha^2$$

Note that one has as well

$$\frac{\alpha^4}{(M_\alpha)^{4\alpha}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(i(-hx)^\alpha E_\alpha(i(-uy)^\alpha E_\alpha(i(-vz)^\alpha E_\alpha(i(-wt)^\alpha(dh)^\alpha(du)^\alpha(dv)^\alpha(dw)^\alpha=\delta_\alpha(x,y,z,t)$$

Inversion Theorem of Quadruple Fractional Laplace Transform

Theorem: Here we recall the fractional quadruple Laplace transform (5) for convenience

$$L_\alpha^4\{f(x,y,z,t)\}=F_\alpha^4(p,q,r,s)(12)$$
$$=\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty E_\alpha(-(px+qy+rz+st)^\alpha)f(x,y,z,t)dx^\alpha dy^\alpha\,dz^\alpha dt^\alpha$$

And its inverse formula define as

$$f(x,y,z,t)=\frac{1}{(M_\alpha)^{4\alpha}}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty E_\alpha((px+qy+rz+st)^\alpha)F_\alpha^4(p,q,r,s)(dp)^\alpha(dq)^\alpha(dr)^\alpha(ds)^\alpha$$
(13)

Proof:

Substituting (12) into (13) and using the formula (11), (9) respectively, we obtain in turn

$$f(x,y,z,t)=\frac{1}{(M_\alpha)^{4\alpha}}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty} E_\alpha\,(px)^\alpha E_\alpha(qy)^\alpha E_\alpha(rz)^\alpha E_\alpha(st)^\alpha(dp)^\alpha(dq)^\alpha\,(dr)^\alpha(ds)^\alpha$$
$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty E_\alpha(-(pj+qk+rl+sm)^\alpha f(j,k,l,m)(dj)^\alpha(dk)^\alpha(dl)^\alpha(dm)^\alpha$$

$$= \frac{1}{(M_\alpha)^{4\alpha}} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty f\,(\beta,\gamma,\psi,\theta)\,(d\beta)^\alpha (d\gamma)^\alpha$$
$$(d\psi)^\alpha (d\theta)^\alpha \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} E_\alpha\,(p^\alpha (x-j)^\alpha)$$
$$E_\alpha(q^\alpha (y-k)^\alpha) E_\alpha(r^\alpha (z-l)^\alpha) E_\alpha(s^\alpha (s-m)^\alpha)$$
$$(dp)^\alpha (dq)^\alpha\,(dr)^\alpha (ds)^\alpha$$
$$= \frac{1}{(M_\alpha)^{4\alpha}} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \frac{(M_\alpha)^{4\alpha}}{\alpha^4} f\,(\beta,\gamma,\psi,\theta)$$
$$\delta_\alpha(j-x,k-y,l-z,m-s)(d\beta)^\alpha$$
$$(d\gamma)^\alpha\,(d\psi)^\alpha (d\theta)^\alpha$$
$$= \frac{1}{\alpha^4} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty f\,(\beta,\gamma,\psi,\theta)\,\delta_\alpha(j$$
$$-x,k-y,$$
$$l-z,m-s)(d\beta)^\alpha (d\gamma)^\alpha\,(d\psi)^\alpha (d\theta)^\alpha$$
$$= f(x,y,z,t)$$

Conclusion

In this present work, fractional quadruple Laplace transform and its inverse are defined, and several properties of fractional quadruple transform have been discussed which are consistent with quadruple Laplace transform when α = 1. More over convolution theorem is presented.

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