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A Common Fixed Point Theorem Using in Fuzzy Metric Space Using Implicit Relation

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ABSTRACT
In this paper, we prove a common fixed point theorem in fuzzy metric space by combining the ideas of point wise R- weak commutativity and reciprocal continuity of mappings satisfying contractive conditions with an implicit relation.

Keywords: *Implicit relation, Common fixed point, R-weakly commuting mappings.*

1. INTRODUCTION

In 1965, Zadeh [11] introduced the concept of Fuzzy set as a new way to represent vagueness in our everyday life. However, when the uncertainty is due to fuzziness rather than randomness, as sometimes in the measurement of an ordinary length, it seems that the concept of a fuzzy metric space is more suitable. We can divide them into following two groups: The first group involves those results in which a fuzzy metric on a set X is treated as a map where X represents the totality of all fuzzy points of a set and satisfy some axioms which are analogous to the ordinary metric axioms. Thus, in such an approach numerical distances are set up between fuzzy objects. On the other hand in second group, we keep those results in which the distance between objects is fuzzy and the objects themselves may or may not be fuzzy. Kramosil et al. (1975)[3] have introduced the concept of fuzzy metric spaces in different ways [1-10]. In this paper, we prove a common fixed point theorem in fuzzy metric space by combining the ideas of point wise R- weak commutativity and reciprocal continuity of mappings satisfying contractive conditions with an implicit relation.

2. Preliminaries:

The concept of triangular norms (t-norms) is originally introduced by Menger in study of statistical metric spaces.

Definition 2.1 (Schweizer & Sklar, 1985)[9] A binary operation $* : [0,1]\times[0,1] \rightarrow [0,1]$ is continuous t-norm if $*$ satisfies the following conditions:

I. $*$ is commutative and associative;
II. $*$ is continuous;
III. $a * 1 = a$ for all $a \in [0,1]$;
IV. $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0,1]$.

Examples of t-norms are: $a*b = min\{a, b\}$, $a*b = ab$ and $a*b = max\{a+b-1,0\}$.

Kramosil et al. (1975)[3] introduced the concept of fuzzy metric spaces as follows:

Definition 2.2 (Kramosil & Michalek, 1975)[3] A 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm, and M is fuzzy sets on $X^2\times[0, \infty)$ satisfying the following conditions for all $x, y, z \in X$ and $s, t > 0$,

I. $M(x, y, 0) = 0$;
II. $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;
III. $M(x, y, t) = M(y, x, t)$;
IV. $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$;
V. $M(x, y, .) : [0, \infty) \rightarrow [0, 1]$ is left continuous.

Then $(X, M, *)$ is called a fuzzy metric space on X. The function $M(x, y, t)$ denote the degree of nearness between x and y w.r.t. t respectively.

Remark 2.3 (Kramosil & Michalek, 1975)[3] In fuzzy metric space $(X, M, *)$, $M(x, y, .)$ is non-decreasing for all $x, y \in X$.

Definition 2.4 (Kramosil & Michalek, 1975)[3] Let $(X, M, *)$ be a fuzzy metric space. Then a sequence $\{x_n\}$ in X is said to be

A. convergent to a point $x \in X$ if, for all $t > 0$,
   a. $lim_{n\to\infty} M(x_n, x, t) = 1$.
   b.
B. Cauchy sequence if, for all $t > 0$ and $p > 0$,
   a. $lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$.

Definition 2.5 (Kramosil & Michalek, 1975) [3] A fuzzy metric space $(X, M, *)$ is said to be complete if and only if every Cauchy sequence in X is convergent.

Definition 2.6 (Vasuki, 1999)[10] A pair of self mappings (A, S) of a fuzzy metric space $(X, M, *)$ is said to be commuting if $M(ASx, SAx, t) = 1$ for all $x \in X$.

Definition 2.7 (Vasuki, 1999)[10] A pair of self mappings (A, S) of a fuzzy metric space $(X, M, *)$ is said to be weakly commuting if $M(ASx, SAx, t) \geq M(Ax, Sx, t)$ for all $x \in X$ and $t > 0$.

Definition 2.8 (Jungck & Rhoades, 2006)[2] A pair of self mappings (A, S) of a fuzzy metric space $(X, M, *)$ is said to be compatible if $lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $lim_{n\to\infty} Ax_n = lim_{n\to\infty} Sx_n = u$ for some $u \in X$.

Definition 2.9(Jungck & Rhoades, 2006)[2] Let $(X, M, *)$ be a fuzzy metric space. A and S be self maps on X. A point $x \in X$ is called a coincidence point of A and S iff $Ax = Sx$. In this case, $w = Ax = Sx$ is called a point of coincidence of A and S.

Definition 2.10 (Jungck & Rhoades, 2006)[2] A pair of self mappings (A, S) of a fuzzy metric space $(X, M, *)$ is said to be weakly compatible if they commute at the coincidence points i.e., if $Au = Su$ for some $u \in X$, then $ASu = SAu$.

It is easy to see that two compatible maps are weakly compatible but converse is not true.

Definition 2.11 (Vasuki, 1999)[10] A pair of self mappings (A, S) of a fuzzy metric space $(X, M, *)$ is said to be pointwise R-weakly commuting if given $x \in X$, there exist $R > 0$ such that $M(ASx, SAx, t) \geq M\left(Ax, Sx, \frac{t}{R}\right)$ for all $t > 0$.

Clearly, every pair of weakly commuting mappings is pointwise R-weakly commuting with $R = 1$.

Definition 2.12 (Pant, 1999)[7] Two mappings A and S of a fuzzy metric space $(X, M, *)$ will be called reciprocally continuous if $ASu_n \to Az, SAu_n \to Sz$, whenever $\{u_n\}$ is a sequence such that $Au_n \to z, Su_n \to z$ for some $z \in X$.

If A and S are both continuous, then they are obviously reciprocally continuous but converse is not true.

Lemma 2.1 (Kramosil & Michalek, 1975)[3] Let $\{u_n\}$ is a sequence in a fuzzy metric space $(X, M, *)$. If there exists a constant $h \in (0,1)$ such that $M(u_n, u_{n+1}, ht) \geq M(u_{n-1}, u_n, t)$, $n = 1, 2, 3, \ldots$ Then $\{u_n\}$ is a Cauchy sequence in X.

3. Main Result:

Let Θ denote the class of those functions $\theta : (0,1]^5 \to [0,1]$ such that θ is continuous and $\theta(x, 1, 1, x, x) = x$.

There are examples of $\theta \in \Theta$:

1. $\theta_1(x_1, x_2, x_3, x_4, x_5) = \min\{x_1, x_2, x_3, x_4, x_5\}$;

2. $\theta_2(x_1, x_2, x_3, x_4, x_5) = \dfrac{x_1(x_1 + x_2 + x_3 + x_4 + x_5)}{(x_1 + x_4 + x_5 + 2)}$;

3. $\theta_3(x_1, x_2, x_3, x_4, x_5) = \sqrt[3]{x_1 x_2 x_3 x_4 x_5}$

Now we prove our main results.

Theorem 3.1 Let *f* and *g* be conditionally reciprocally continuous self-mappings of a complete fuzzy metric space $(X, M, *)$ satisfying the conditions:

(3.1) $f(X) \subseteq g(X)$;

(3.2) for any $x, y \in X$, $t > 0$ and $k \in (0,1)$ such that:

$$M(fx, fy, kt) \geq \min\{M(gx, gy, t), M(gx, fy, 2t), M(fx, gx, t), M(fx, gy, t), M(fy, gy, t)\};$$

If *f* and *g* are either compatible or *g*- compatible or *f*- compatible then *f* and *g* have a unique common fixed point.

Proof. Let x_0 be any point in *X*. Then as $f(X) \subseteq g(X)$, there exist a sequence of points $\{x_n\}$ such that $f(x_n) = g(x_{n+1})$.

Also, define a sequence $\{y_n\}$ in *X* as $y_n = f(x_n) = g(x_{n+1})$. (3.3)

Now, we show that $\{y_n\}$ is a Cauchy sequence in *X*. For proving this, by (3.2), we have

$$
\begin{aligned}
M(y_n, y_{n+1}, kt) &= M(fx_n, fx_{n+1}, kt) \\
&\geq \min\left\{\begin{matrix} M(gx_n, gx_{n+1}, t), M(gx_n, fx_{n+1}, 2t), M(fx_n, gx_n, t), \\ M(fx_n, gx_{n+1}, t), M(fx_{n+1}, gx_{n+1}, t) \end{matrix}\right\} \\
&= \min\left\{\begin{matrix} M(y_{n-1}, y_n, t), M(y_{n-1}, y_{n+1}, 2t), M(y_n, y_{n-1}, t), \\ M(y_n, y_n, t), M(y_{n+1}, y_n, t) \end{matrix}\right\} \\
&\geq \min\left\{\begin{matrix} M(y_{n-1}, y_n, t), M(y_{n-1}, y_n, t), M(y_n, y_{n+1}, t), \\ M(y_n, y_{n-1}, t), 1, M(y_{n+1}, y_n, t) \end{matrix}\right\} \\
&= \min\{M(y_{n-1}, y_n, t), M(y_n, y_{n+1}, t)\}
\end{aligned}
$$

$$M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t)$$

Then, by lemma 2.2, $\{y_n\}$ is a Cauchy sequence in *X*. As, *X* is complete, there exist a point *z* in *X* such that $\lim_{n\to\infty} y_n = z$. Therefore, by (3.3), we have $\lim_{n\to\infty} y_n = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_{n+1}) = z$. Since *f* and *g* be conditionally reciprocally continuous and $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = z$, there exist a sequence $\{s_n\}$ satisfying $\lim_{n\to\infty} f(s_n) = \lim_{n\to\infty} g(s_n) = u(say)$ such that $\lim_{n\to\infty} fg(s_n) = fu$ and $\lim_{n\to\infty} gf(s_n) = gu$. Since, $f(X) \subseteq g(X)$, for each s_n, there exist z_n in X such that $fs_n = gz_n$. Thus,

$\lim_{n\to\infty} f(s_n) = \lim_{n\to\infty} g(s_n) = \lim_{n\to\infty} g(z_n) = u$. By using (3.2), we get

$$M(fs_n, fz_n, kt) \geq \min\left\{\begin{matrix} M(gs_n, gz_n, t), M(gs_n, fz_n, 2t), M(fs_n, gs_n, t), \\ M(fs_n, gz_n, t), M(fz_n, gz_n, t) \end{matrix}\right\}$$

$n \to \infty$

$$
\begin{aligned}
M(u, \lim_{n\to\infty} fz_n, kt) &\geq \min\left\{\begin{matrix} M(u, u, t), M(u, fz_n, 2t), M(u, u, t), \\ M(u, u, t), M(fz_n, u, t) \end{matrix}\right\} \\
&\geq \min\left\{\begin{matrix} 1, M(u, \lim_{n\to\infty} fz_n, t), 1, \\ 1, M(\lim_{n\to\infty} fz_n, u, t) \end{matrix}\right\} \\
&= M(u, \lim_{n\to\infty} fz_n, t)
\end{aligned}
$$

this gives, $\lim_{n\to\infty} f(z_n) = u$. Hence, $\lim_{n\to\infty} f(s_n) = \lim_{n\to\infty} g(s_n) = \lim_{n\to\infty} g(z_n) = \lim_{n\to\infty} f(z_n) = u$ (3.4)

Suppose that *f* and *g* are compatible mappings. Then $\lim_{n\to\infty} M(fg(s_n), gf(s_n), t) = 1$, that is, $\lim_{n\to\infty} fg(s_n) = \lim_{n\to\infty} gf(s_n)$, this gives, $fu = gu$. Also, $fgu = ffu = fgu = gfu$. Using (3.2), we get

$$M(fu, ffu, kt) \geq \min\begin{Bmatrix} M(gu, gfu, t), M(gu, ffu, 2t), M(fu, gu, t), \\ M(fu, gfu, t), M(ffu, gfu, t) \end{Bmatrix}$$

$$\geq \min\begin{Bmatrix} M(fu, ffu, t), M(fu, ffu, t), M(fu, fu, t), \\ M(fu, ffu, t), M(ffu, ffu, t) \end{Bmatrix}$$

$$= M(fu, ffu, t)$$

That is $fu = ffu$. Hence, $fu = ffu = gfu$ and fu is a common fixed point of f and g.

Now, Suppose that f and g are g-compatible mappings. Then $\lim_{n\to\infty} M\left(ff(s_n), gf(s_n), t\right) = 1$, that is, $\lim_{n\to\infty} ff(s_n) = \lim_{n\to\infty} gf(s_n) = gu$. Using (3.2), we get

$$M(fu, ffs_n, kt) \geq \min\left\{M(gu, gfs_n, t), M(gu, ffs_n, 2t), M(fu, gu, t), M(fu, gfs_n, t), M(ffs_n, gfs_n, t)\right\}$$

$n \to \infty$

$$M(fu, gu, kt) \geq \min\left\{M(gu, gu, t), M(gu, gu, 2t), M(fu, gu, t), M(fu, gu, t), M(gu, gu, t)\right\}$$

$$M(fu, gu, kt) \geq M(fu, gu, t)$$

This gives, $fu = gu$. Also, $fgu = ffu = fgu = gfu$. Using (3.2), we get

$$M(fu, ffu, kt) \geq \min\left\{M(gu, gfu, t), M(gu, ffu, 2t), M(fu, gu, t), M(fu, gfu, t), M(ffu, gfu, t)\right\}$$

$$\geq \min\left\{M(fu, ffu, t), M(fu, ffu, t), M(fu, fu, t), M(fu, ffu, t), M(ffu, ffu, t)\right\}$$

$$= M(fu, ffu, t)$$

That is $fu = ffu$. Hence, $fu = ffu = gfu$ and fu is a common fixed point of f and g.

Finally, Suppose that f and g are f-compatible mappings. Then $\lim_{n\to\infty} M\left(fg(z_n), gg(z_n), t\right) = 1$, that is, $\lim_{n\to\infty} fg(z_n) = \lim_{n\to\infty} gg(z_n)$. Also, $\lim_{n\to\infty} gf(s_n) = \lim_{n\to\infty} gg(z_n) = gu$.

Therefore, $\lim_{n\to\infty} fg(z_n) = \lim_{n\to\infty} gg(z_n) = gu$.

Using (3.2), we get

$$M(fu, fgz_n, kt) \geq \min\begin{Bmatrix} M(gu, ggz_n, t), M(gu, fgz_n, 2t), \\ M(fu, gu, t), M(fu, ggz_n, t), M(fgz_n, ggz_n, t) \end{Bmatrix}$$

$n \to \infty$

$$M(fu, gu, kt) \geq \min\begin{Bmatrix} M(gu, gu, t), M(gu, gu, 2t), \\ M(fu, gu, t), M(fu, gu, t), M(gu, gu, t) \end{Bmatrix}$$

$$M(fu, gu, kt) \geq M(fu, gu, t)$$

This gives, $fu = gu$. Also, $fgu = ffu = fgu = gfu$. Using (3.2), we get

$$M(fu, ffu, kt) \geq \min\begin{Bmatrix} M(gu, gfu, t), M(gu, ffu, 2t), \\ M(fu, gu, t), M(fu, gfu, t), M(ffu, gfu, t) \end{Bmatrix}$$

$$M(fu, ffu, kt) \geq \min\begin{Bmatrix} M(fu, ffu, t), M(fu, ffu, 2t), M(fu, fu, t), \\ M(fu, ffu, t), M(ffu, ffu, t) \end{Bmatrix}$$

$$M(fu, ffu, kt) \geq M(fu, ffu, t)$$

That is $fu = ffu$. Hence, $fu = ffu = gfu$ and fu is a common fixed point of f and g.

Uniqueness of the common fixed point theorem follows easily in each of the three cases.

Theorem 3.2 Let *f* and *g* be non-compatible self-mappings of a fuzzy metric space $(X, M, *)$ satisfying the conditions:

(3.4) $f(X) \subseteq g(X)$;

(3.5) for all $k \in (0,1)$ such that:

$$M(fx, fy, kt) \geq \min\left\{M(gx, gy, t), M(gx, fy, 2t), M(fx, gx, t), M(fx, gy, t), M(fy, gy, t)\right\};$$

(3.6) $M(fx, ffx, t) > M(gx, ggx, t)$ whenever $gx \neq ggx$ for all $x, y \in X$ and $t > 0$.

Suppose *f* and *g* be conditionally reciprocally continuous If *f* and *g* are either *g*- compatible or *f*- compatible then *f* and *g* have fixed point.

Proof: Since *f* and *g* are non-compatible maps, there exists a sequence $\{x_n\}$ in X such that $fx_n \to z$ and $gx_n \to z$ for some z in X as $n \to \infty$ but either $\lim_{n\to\infty} M(fgx_n, gfx_n, t) \neq 1$ or the limit does not exist. Also, since f and g be conditionally reciprocally continuous and $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = z$, there exist a sequence $\{y_n\}$ satisfying $\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} g(y_n) = u(say)$ such that $\lim_{n\to\infty} fg(y_n) = fu$ and $\lim_{n\to\infty} gf(y_n) = gu$. Since, $f(X) \subseteq g(X)$, for each y_n, there exist z_n in X such that $fy_n = gz_n$. Thus, $\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} g(y_n) = \lim_{n\to\infty} g(z_n) = u$.

By using (3.5), we get

$$M(fy_n, fz_n, kt) \geq \min\begin{Bmatrix} M(gy_n, gz_n, t), M(gy_n, fz_n, 2t), M(fy_n, gy_n, t), \\ M(fy_n, gz_n, t), M(fz_n, gz_n, t) \end{Bmatrix}$$

$n \to \infty$

$$M(u, \lim_{n\to\infty} fz_n, kt) \geq \min\begin{Bmatrix} M(u,u,t), M(u, \lim_{n\to\infty} fz_n, 2t), M(u,u,t), \\ M(u,u,t), M(\lim_{n\to\infty} fz_n, u, t) \end{Bmatrix}$$

$$\geq \min\left\{1, M(u, \lim_{n\to\infty} fz_n, t), 1, 1, M(\lim_{n\to\infty} fz_n, u, t)\right\}$$

$$= M(u, \lim_{n\to\infty} fz_n, t)$$

this gives, $\lim_{n\to\infty} f(z_n) = u$. Therefore, we have $\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} g(y_n) = \lim_{n\to\infty} g(z_n) = \lim_{n\to\infty} f(z_n) = u$.

Now, Suppose that *f* and *g* are *g*-compatible mappings. Then $\lim_{n\to\infty} M\left(ff(y_n), gf(y_n), t\right) = 1$, that is, $\lim_{n\to\infty} ff(y_n) = \lim_{n\to\infty} gf(y_n) = gu$. Using (3.5), we get

$$M(fu, ffy_n, kt) \geq \min\left\{M(gu, gfy_n, t), M(gu, ffy_n, 2t), M(fu, gu, t), M(fu, gfy_n, t), M(ffy_n, gfy_n, t)\right\}$$

$n \to \infty$

$$M(fu, gu, kt) \geq \min\left\{M(gu, gu, t), M(gu, gu, 2t), M(fu, gu, t), M(fu, gu, t), M(gu, gu, t)\right\}$$

$$M(fu, gu, kt) \geq M(fu, gu, t)$$

this gives, $fu = gu$. Also, $fgu = ffu = fgu = gfu$. If $fu \neq ffu$, using (3.6), we get $M(fu, ffu, t) > M(gu, ggu, t) = M(fu, ffu, t)$, a contradiction. Hence, $fu = ffu = gfu$ and fu is a common fixed point of f and g.

Finally, Suppose that *f* and *g* are *f*-compatible mappings. Then $\lim_{n\to\infty} M\left(fg(z_n), gg(z_n), t\right) = 1$, that is, $\lim_{n\to\infty} fg(z_n) = \lim_{n\to\infty} gg(z_n)$. Also, $\lim_{n\to\infty} gf(y_n) = \lim_{n\to\infty} gg(z_n) = gu$.

Therefore, $\lim_{n\to\infty} fg(z_n) = \lim_{n\to\infty} gg(z_n) = gu$.

Using (3.5), we get

$$M(fu, fgz_n, kt) \geq \min\begin{Bmatrix} M(gu, ggz_n, t), M(gu, fgz_n, 2t), \\ M(fu, gu, t), M(fu, ggz_n, t), M(fgz_n, ggz_n, t) \end{Bmatrix}$$

$n \to \infty$

$$M(fu, gu, kt) \geq \min\begin{Bmatrix} M(gu, gu, t), M(gu, gu, 2t), \\ M(fu, gu, t), M(fu, gu, t), M(gu, gu, t) \end{Bmatrix}$$

$$M(fu, gu, kt) \geq M(fu, gu, t)$$

This gives, $fu = gu$. Also, $fgu = ffu = fgu = gfu$. If $fu \neq ffu$, using (3.6), we get $M(fu, ffu, t) > M(gu, ggu, t) = M(fu, ffu, t)$, a contradiction. Hence, $fu = ffu = gfu$ and fu is a common fixed point of f and g.

Theorem 3.3 Let f and g be non-compatible self-mappings of a fuzzy metric space $(X, M, *)$ satisfying the conditions:

(3.7) $f(X) \subseteq g(X)$;

(3.8) for all $k \in (0,1)$ such that:

$$M(fx, fy, kt) \geq \min\{M(gx, gy, t), M(gx, fy, 2t), M(fx, gx, t), M(fx, gy, t), M(fy, gy, t)\};$$

(3.9) $$M(fx, f^2x, t) > \max\begin{Bmatrix} M(gx, gfx, t), M(fx, gx, t), M(f^2x, gfx, t), \\ M(fx, gfx, t), M(gx, f^2x, t) \end{Bmatrix}$$

Whenever $fx \neq f^2x$ for all $x, y \in X$ and $t > 0$.

Suppose f and g be conditionally reciprocally continuous If f and g are either g- compatible or f- compatible then f and g common fixed point.

Proof: Since f and g are non-compatible maps, there exists a sequence $\{x_n\}$ in X such That $fx_n \to z$ and $gx_n \to z$ for some z in X as $n \to \infty$ but either $\lim_{n\to\infty} M(fgx_n, gfx_n, t) \neq 1$ or the limit does not exist. Also, since f and g be conditionally reciprocally continuous and $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = z$, there exist a sequence $\{y_n\}$ satisfying $\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} g(y_n) = u(say)$ such that $\lim_{n\to\infty} fg(y_n) = fu$ and $\lim_{n\to\infty} gf(y_n) = gu$. Since, $f(X) \subseteq g(X)$, for each y_n, there exist z_n in X such that $fy_n = gz_n$. Thus, $\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} g(y_n) = \lim_{n\to\infty} g(z_n) = u$. By using (3.8), we get

$$M(fy_n, fz_n, kt) \geq \min\begin{Bmatrix} M(gy_n, gz_n, t), M(gy_n, fz_n, 2t), M(fy_n, gy_n, t), \\ M(fy_n, gz_n, t), M(fz_n, gz_n, t) \end{Bmatrix}$$

$n \to \infty$

$$M(u, \lim_{n\to\infty} fz_n, kt) \geq \min\begin{Bmatrix} M(u, u, t), M(u, \lim_{n\to\infty} fz_n, 2t), M(u, u, t), \\ M(u, u, t), M(\lim_{n\to\infty} fz_n, u, t) \end{Bmatrix}$$

$$\geq \min\left\{1, M(u, \lim_{n\to\infty} fz_n, t), 1, 1, M(\lim_{n\to\infty} fz_n, u, t)\right\}$$

$$= M(u, \lim_{n\to\infty} fz_n, t)$$

This gives, $\lim_{n\to\infty} f(z_n) = u$. Therefore, we have $\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} g(y_n) = \lim_{n\to\infty} g(z_n) = \lim_{n\to\infty} f(z_n) = u$.

Now, Suppose that f and g are g-compatible mappings. Then $\lim_{n\to\infty} M\left(ff(y_n), gf(y_n), t\right) = 1$, that is, $\lim_{n\to\infty} ff(y_n) = \lim_{n\to\infty} gf(y_n) = gu$. Using (3.8), we get

$M(fu, ffy_n, kt) \geq \min\{M(gu, gfy_n, t), M(gu, ffy_n, 2t), M(fu, gu, t), M(fu, gfy_n, t), M(ffy_n, gfy_n, t)\}$

$n \to \infty$

$M(fu, gu, kt) \geq \min\{M(gu, gu, t), M(gu, gu, 2t), M(fu, gu, t), M(fu, gu, t), M(gu, gu, t)\}$

$M(fu, gu, kt) \geq M(fu, gu, t)$

This gives, $fu = gu$. Also, $fgu = ffu = fgu = gfu$. If $fu \neq ffu$, using (3.9), we get

$$M(fu, f^2u, t) > \max\begin{Bmatrix} M(gu, gfu, t), M(fu, gu, t), M(f^2u, gfu, t), \\ M(fu, gfu, t), M(gu, f^2u, t) \end{Bmatrix}$$

$$M(fu, f^2u, t) > \max\begin{Bmatrix} M(fu, ffu, t), M(fu, fu, t), M(f^2u, ffu, t), \\ M(fu, ffu, t), M(fu, f^2u, t) \end{Bmatrix}$$

$M(fu, f^2u, t) > M(fu, f^2u, t)$

a contradiction. Hence, $fu = ffu = gfu$ and fu is a common fixed point of f and g.

Finally, Suppose that f and g are f-compatible mappings. Then $\lim_{n\to\infty} M(fg(z_n), gg(z_n), t) = 1$, that is, $\lim_{n\to\infty} fg(z_n) = \lim_{n\to\infty} gg(z_n)$. Also, $\lim_{n\to\infty} gf(y_n) = \lim_{n\to\infty} gg(z_n) = gu$.

Therefore, $\lim_{n\to\infty} fg(z_n) = \lim_{n\to\infty} gg(z_n) = gu$.

Using (3.8), we get

$$M(fu, fgz_n, kt) \geq \min\begin{Bmatrix} M(gu, ggz_n, t), M(gu, fgz_n, 2t), \\ M(fu, gu, t), M(fu, ggz_n, t), M(fgz_n, ggz_n, t) \end{Bmatrix}$$

$n \to \infty$

$$M(fu, gu, kt) \geq \min\begin{Bmatrix} M(gu, gu, t), M(gu, gu, 2t), \\ M(fu, gu, t), M(fu, gu, t), M(gu, gu, t) \end{Bmatrix}$$

$M(fu, gu, kt) \geq M(fu, gu, t)$

This gives, $fu = gu$. Also, $fgu = ffu = fgu = gfu$. If $fu \neq ffu$, using (3.9), we get

$$M(fu, f^2u, t) > \max\begin{Bmatrix} M(gu, gfu, t), M(fu, gu, t), M(f^2u, gfu, t), \\ M(fu, gfu, t), M(gu, f^2u, t) \end{Bmatrix}$$

$$M(fu, f^2u, t) > \max\begin{Bmatrix} M(fu, ffu, t), M(fu, fu, t), M(f^2u, ffu, t), \\ M(fu, ffu, t), M(fu, f^2u, t) \end{Bmatrix}$$

$M(fu, f^2u, t) > M(fu, f^2u, t)$

A contradiction Hence, $fu = ffu = gfu$ and fu is a common fixed point of f and g.

If we take θ as θ_1, θ_2, θ_3, then we get the following corollaries:

Corollary 4.3 Let f and g be weakly reciprocally continuous non-compatible self-mappings of a fuzzy metric space $(X, M, *)$ satisfying the conditions:

(4.1) $f(X) \subset g(X)$;

(4.7) $\int_0^{M(fx,fy,t)} \psi(s)ds \geq \int_0^{M(gx,gy,t)} \psi(s)ds$;

(4.8) $\int_0^{M(fx,f^2x,t)} \psi(s)ds > \int_0^{\min\begin{Bmatrix} M(gx,gfx,t), M(fx,gx,t), M(f^2x,gfx,t), \\ M(fx,gfx,t), M(gx,f^2x,t) \end{Bmatrix}} \psi(s)ds$

Whenever $fx \neq f^2x$ for all $x, y \in X$, $t > 0$, where $\psi : \mathbb{R}^+ \to \mathbb{R}$ is a Lebesgue integrable mapping which is summable nonnegative and such that $\int_0^{\varepsilon} \psi(s)ds > 0$ for each $\varepsilon > 0$.

If *f* and *g* are *R*-weakly commuting of type *(A_g)* or R-weakly commuting of type *(A_f)* or *R*-weakly commuting of type (*P*) then *f* and *g* have a unique common fixed point.

Corollary 4.4 Let *f* and *g* be weakly reciprocally continuous non-compatible self-mappings of a fuzzy metric space $(X, M, *)$ satisfying the conditions:

(4.1) $f(X) \subset g(X)$;

(4.7) $\int_0^{M(fx,fy,t)} \psi(s)ds \geq \int_0^{M(gx,gy,t)} \psi(s)ds$;

(4.8) $\int_0^{M(fx,f^2x,t)} \psi(s)ds > \int_0^{\frac{M(gx,gfx,t)\left(M(gx,gfx,t)+M(fx,gx,t)+M(f^2x,gfx,t)+M(fx,gfx,t)+M(gx,f^2x,t)\right)}{M(gx,gfx,t)+M(fx,gfx,t)+M(gx,f^2x,t)+2}} \psi(s)ds$

Whenever $fx \neq f^2x$ for all $x, y \in X$, $t > 0$, where $\psi : \mathbb{R}^+ \to \mathbb{R}$ is a Lebesgue integrable mapping which is summable nonnegative and such that $\int_0^{\varepsilon} \psi(s)ds > 0$ for each $\varepsilon > 0$.

If *f* and *g* are *R*-weakly commuting of type *(A_g)* or R-weakly commuting of type *(A_f)* or *R*-weakly commuting of type (*P*) then *f* and *g* have a unique common fixed point.

Corollary 4.5 Let *f* and *g* be weakly reciprocally continuous non-compatible self-mappings of a fuzzy metric space $(X, M, *)$ satisfying the conditions:

(4.1) $f(X) \subset g(X)$;

(4.7) $\int_0^{M(fx,fy,t)} \psi(s)ds \geq \int_0^{M(gx,gy,t)} \psi(s)ds$;

(4.8) $\int_0^{M(fx,f^2x,t)} \psi(s)ds > \int_0^{\sqrt[3]{M(gx,gfx,t).M(fx,gx,t).M(f^2x,gfx,t).M(fx,gfx,t).M(gx,f^2x,t)}} \psi(s)ds$

whenever $fx \neq f^2x$ for all $x, y \in X$, $t > 0$, where $\psi : \mathbb{R}^+ \to \mathbb{R}$ is a Lebesgue integrable mapping which is summable nonnegative and such that $\int_0^{\varepsilon} \psi(s)ds > 0$ for each $\varepsilon > 0$.

If *f* and *g* are *R*-weakly commuting of type *(A_g)* or R-weakly commuting of type *(A_f)* or *R*-weakly commuting of type (*P*) then *f* and *g* have a unique common fixed point.

Let Δ denote the class of those functions $\delta : (0,1]^5 \to [0,1]$ such that δ is continuous and $\delta(x,1,x,1) = x$.

There are examples of $\delta \in \Delta$:

1. $\delta_1(x_1, x_2, x_3, x_4) = \min\{x_1, x_2, x_3, x_4\}$;
2. $\delta_2(x_1, x_2, x_3, x_4) = \sqrt{x_1 x_2 x_3 x_4}$;
3. $\delta_3(x_1, x_2, x_3, x_4) = \min\left\{\sqrt{x_1 x_2}, \sqrt{x_3 x_4}\right\}$.

Now we prove our main results.

Theorem 4.5 Let *f* and *g* be weakly reciprocally continuous non-compatible self-mappings of a fuzzy metric space $(X, M, *)$ satisfying the conditions:

(4.1) $f(X) \subset g(X)$;

(4.7) $\int_0^{M(fx,fy,t)} \psi(s)ds \geq \int_0^{M(gx,gy,t)} \psi(s)ds$;

(4.8) $\int_0^{M(fx,f^2x,t)} \psi(s)ds > \int_0^{\delta\left(M(gx,gfx,t),M(fx,gx,t),M(fx,gfx,t),M(f^2x,gfx,t)\right)} \psi(s)ds$

whenever $fx \neq f^2x$ for all $x, y \in X$, $t > 0$ and for some $\delta \in \Delta$ where $\psi : ¡^+ \to ¡$ is a Lebesgue integrable mapping which is summable nonnegative and such that $\int_0^{\varepsilon} \psi(s)ds > 0$ for each $\varepsilon > 0$.

If *f* and *g* are *R*-weakly commuting of type *(A_g)* or R-weakly commuting of type *(A_f)* or *R*-weakly commuting of type (*P*) then *f* and *g* have a unique common fixed point.

Proof. Since *f* and *g* are non-compatible maps, there exists a sequence $\{x_n\}$ in *X* such that $f x_n \to z$ and $gx_n \to z$ for some *z* in *X* as $n\to\infty$

but either $\lim_{n\to\infty} M(fgx_n, gfx_n, t) \neq 1$ or the limit does not exist. Since $f(X) \subset g(X)$, for each $\{x_n\}$ there exists $\{y_n\}$ in *X* such that $f x_n = gy_n$. Thus $f x_n \to z$, $gx_n \to z$ and $gy_n \to z$ as $n\to\infty$. By virtue of this and using (4.7) we obtain

$$\int_0^{M(fx_n,fy_n,t)} \psi(s)ds \geq \int_0^{M(gx_n,gy_n,t)} \psi(s)ds$$

$$n \to \infty$$

$$\int_0^{M(z,\lim_{n\to\infty} fy_n,t)} \psi(s)ds \geq \int_0^{M(z,z,t)} \psi(s)ds$$

which implies that, $fy_n \to z$ as $n\to\infty$. Therefore, we have $fx_n \to z$, $gx_n \to z$, $gy_n \to z$, $fy_n \to z$. Suppose that *f* and *g* are *R*-weakly commuting of type (A_g). Then weak reciprocal continuity of *f* and *g* implies that $fgx_n \to f z$ or $g fx_n \to gz$. Similarly, $fgy_n \to fz$ or $gfy_n \to gz$. Let us first assume that $gfy_n \to gz$. Then *R*-weak commutativity of type *(A_g)* of *f* and *g* yields

$$M\left(ffy_n, gfy_n, t\right) \geq M\left(fy_n, gy_n, t/R\right)$$

$$n \to \infty$$

$$\lim_{n\to\infty} M\left(ffy_n, gz, t\right) \geq M\left(z, z, t/R\right) = 1$$

This gives, $f f y_n \to gz$. Using (4.7), we get

$$\int_0^{M(ffy_n,fz,kt)} \psi(s)ds \geq \int_0^{M(gfy_n,gz,t)} \psi(s)ds$$

$$n \to \infty$$

$$\int_0^{M(gz,fz,kt)} \psi(s)ds \geq \int_0^{M(gz,gz,t)} \psi(s)ds$$

This implies that $f z = gz$. Again, by virtue of *R*-weak commutativity of type (A_g),

$M\left(ffz, gfz, t\right) \geq M\left(fz, gz, t/R\right) = 1$. This yields $f f z = g f z$ and $f f z = f gz = g f z = ggz$. If $fz \neq ffz$ then by using (4.8), we get

$$\begin{aligned}\int_0^{M(fz,f^2z,t)} \psi(s)ds &> \int_0^{\delta\left(M(gz,gfz,t),M(fz,gz,t),M(fz,gfz,t),M(f^2z,gfz,t)\right)} \psi(s)ds \\ &= \int_0^{\delta\left(M(fz,f^2z,t),M(fz,fz,t),M(fz,f^2z,t),M(f^2z,f^2z,t)\right)} \psi(s)ds \\ &= \int_0^{\delta\left(M(fz,f^2z,t),1,M(fz,f^2z,t),1\right)} \psi(s)ds \\ &= \int_0^{M(fz,f^2z,t)} \psi(s)ds\end{aligned}$$

a contradiction. Hence $f z = f f z = g f z$ and $f z$ is a common fixed point of *f* and *g*.

Similarly, we can prove, if $fgy_n \to fz,$ then again fz is a common fixed point of f and g. Proof is similar if f and g are R-weakly commuting of type (A_f) or *(P)*. Uniqueness of the common fixed point theorem follows easily in each of the two cases.

If we take δ as $\delta_1, \delta_2, \delta_3$ then we get the following corollaries:

Corollary 4.5 Let f and g be weakly reciprocally continuous non-compatible self-mappings of a fuzzy metric space $(X, M, *)$ satisfying the conditions:

(4.1) $f(X) \subset g(X)$;

(4.7) $\int_0^{M(fx,fy,t)} \psi(s)ds \geq \int_0^{M(gx,gy,t)} \psi(s)ds$;

(4.8) $\int_0^{M(fx,f^2x,t)} \psi(s)ds > \int_0^{\min\{M(gx,gfx,t), M(fx,gx,t), M(fx,gfx,t), M(f^2x,gfx,t)\}} \psi(s)ds$

whenever $fx \neq f^2x$ for all $x, y \in X$, $t > 0$ where $\psi : \mathbb{R}^+ \to \mathbb{R}$ is a Lebesgue integrable mapping which is summable nonnegative and such that $\int_0^{\varepsilon} \psi(s)ds > 0$ for each $\varepsilon > 0$.

If f and g are R-weakly commuting of type *(A_g)* or R-weakly commuting of type *(A_f)* or R-weakly commuting of type (P) then f and g have a unique common fixed point.

Corollary 4.5 Let f and g be weakly reciprocally continuous non-compatible self-mappings of a fuzzy metric space $(X, M, *)$ satisfying the conditions:

(4.1) $f(X) \subset g(X)$;

(4.7) $\int_0^{M(fx,fy,t)} \psi(s)ds \geq \int_0^{M(gx,gy,t)} \psi(s)ds$;

(4.8) $\int_0^{M(fx,f^2x,t)} \psi(s)ds > \int_0^{\sqrt{M(gx,gfx,t).M(fx,gx,t).M(fx,gfx,t).M(f^2x,gfx,t)}} \psi(s)ds$

whenever $fx \neq f^2x$ for all $x, y \in X$, $t > 0$ where $\psi : \mathbb{R}^+ \to \mathbb{R}$ is a Lebesgue integrable mapping which is summable nonnegative and such that $\int_0^{\varepsilon} \psi(s)ds > 0$ for each $\varepsilon > 0$.

If f and g are R-weakly commuting of type *(A_g)* or R-weakly commuting of type *(A_f)* or R-weakly commuting of type (P) then f and g have a unique common fixed point.

Corollary 4.5 Let f and g be weakly reciprocally continuous non-compatible self-mappings of a fuzzy metric space $(X, M, *)$ satisfying the conditions:

(4.1 $f(X) \subset g(X)$;

(4.7) $\int_0^{M(fx,fy,t)} \psi(s)ds \geq \int_0^{M(gx,gy,t)} \psi(s)ds$;

(4.8) $\int_0^{M(fx,f^2x,t)} \psi(s)ds > \int_0^{\min\left\{\sqrt{M(gx,gfx,t).M(fx,gx,t)}, \sqrt{M(fx,gfx,t).M(f^2x,gfx,t)}\right\}} \psi(s)ds$

whenever $fx \neq f^2x$ for all $x, y \in X$, $t > 0$ where $\psi : \mathbb{R}^+ \to \mathbb{R}$ is a Lebesgue integrable mapping which is summable nonnegative and such that $\int_0^{\varepsilon} \psi(s)ds > 0$ for each $\varepsilon > 0$.

If f and g are R-weakly commuting of type *(A_g)* or R-weakly commuting of type *(A_f)* or R-weakly commuting of type (P) then f and g have a unique common fixed point.

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