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Analytical solutions of (2+1)-dimensional Burgers' equation with damping term by HPM,ADM and DTM

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Abstract: We have employed Analytical solutions of $(2+1)$-dimensional Burgers' equation with damping term by HPM,ADM and DTM as.

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In the past few decades, traditional integral transform methods such as Fourier and Laplace transforms have commonly been used to solve engineering problems. These methods transform differential equations into algebraic equations which are easier to deal with. However, these integral transform methods are more complex and difficult when applying to nonlinear problems.

The HPM, proposed first by He[2,3], for solving the differential and integral equations, linear and nonlinear, has been the subject of extensive analytical and numerical studies. The method, which is a coupling of the traditional perturbation method and homotopy in topology, deforms continuously to a simple problem which is easily solved. This method, which does not require a small parameter in an equation, has a significant advantage in that it provides an analytical approximate solution to a wide range of nonlinear problems in applied sciences.

The HPM is applied to Volterra's integro-differential equation [4], to nonlinear oscillators [5], bifurcation of nonlinear problems [6], bifurcation of delay-differential equations [7], nonlinear wave equations [8], boundary value problems [9], quadratic Riccati differential equation of fractional order [1], and to other fields [10-18]. This HPM yields a very rapid convergence of the solution series in most cases, usually only a few iterations leading to very accurate solutions. Thus He's HPM is a universal one which can solve various kinds of nonlinear equations.

Adomian decomposition method(ADM), which was introduced by Adomian[19], is a semi numerical technique for solving linear and nonlinear differential equations by generating a funtional series solution in a very efficient manner. The method has many advantages: it solves the problem directly without the need for linearization, perturbation, or any other transformation; it converges very rapidly and is highly accurate.

Differential transform method (DTM),which was first applied in the engineering field by Zhou[8],has many advantages :it solves the problem directly without the need for linearization,pertuburation,or any other transformation.DTM is based on the Taylors series expansion.It is different from the traditional high order Taylor series method,which needs symbolic computation of the necessary derivatives of the data functions.Taylor series method computationally takes a long time for larger orders while DTM method reduces the size of the computational domain,without massive computations and restrictive assumpions,and is easily applicable to various physical problems .The method and related theorems are well adressed in [9,10].Let us consider the (2+1)-dimensional Burgers' equation with damping term as,

$$u_t = u_x u_y + u + u_{xy}, \tag{1}$$

under the initial condition

$$u(x, y, 0) = u_0(x, y). \tag{2}$$

Burger's equation is the simplest second order NLPDE which balances the effect of nonlinear convection and the linear diffusion.In this work ,we have employed the Homotopy Perturbation Method(HPM),DTM and ADM to solve the (2+1)dimensional Burger's equation with damping term.

1.1 Homotopy Perturbation Method (HPM)

To describe the HPM, consider the following general nonlinear differential equation

$$A(u) - f(r) = 0, r\epsilon\Omega, \tag{3}$$

under the boundary condition

$$B(u, \frac{\partial u}{\partial n}) = 0, r\epsilon\partial\Omega, \tag{4}$$

Where A is a general differential operator, B is a boundary operator,f(r)is a known analytic function and $\partial\Omega$ is a boundary of the domain Ω.The operator A can be divided into two parts L and N, Where L is a linear opertor while N is a nonlinear operator.Then Equation(3)can be rewritten as

$$L(u) + N(u) - f(r) = 0, \tag{5}$$

Using the homotopy tecnique, we construct a homotopy:

$$V(r, p) : \Omega X[0, 1] \to R$$

Which satisfies

$$H(V,p) = (1-p)[L(V) - L(u_0)] + p[A(V) - f(r)]$$

or

$$H(V,p) = L(V) - L(u_0) + pL(u_0) + p[N(V) - f(r)], r\epsilon\Omega, p\epsilon[0,1], \quad (6)$$

Where $p\epsilon[0,1]$ is an embedding parameter, u_0 is the initial approximation of Equation(3) which satisfies the boundary conditions.Obviously,considering Equation(6),we will have

$$H(V,0) = L(V) - L(u_0) = 0,$$

$$H(V,1) = A(V) - f(r) = 0, \quad (7)$$

changing the process of p from zero to unity is just that V(r,p)from $u_0(r)$ to u(r).In topology,this is called the deformation also A(V)-f(r) and L(u)are called as homotopy .The homotopy perturbation method uses the homotopy parameter p as an expanding parameter [23-25]to obtain

$$V = v_0 + pv_1 + p^2v_2 + p^3v_3 + ... = \sum_{n=0}^{\infty} p^n v_n. \quad (8)$$

p $\rightarrow$1 results the appropriate solution of equation(3)as

$$u = \lim_{p\rightarrow 1} V = v_0 + v_1 + v_2 + ... = \sum_{n=0}^{\infty} v_n. \quad (9)$$

A comparision of like powers of p gives the solutions of various orders series(9)is convergent for most of the cases.However convergence rate depends on the nonlinear,$N(V)$ He[25]suggested the following opinions:

1.The second derivative of N(V)with respect to v must be small as the parameter p may be relatively large.

2.The norm of $L^{-1}\dfrac{\partial N}{\partial u}$ must be smaller than one so that the series converges. In this section, we describe the above method by the following example to validate the efficiency of the HPM.

Example:1

Consider the (2+1)-dimensional Burgers' equation with daming as,

$$u_t = u_{xy} + u_x u_y + u, \quad (10)$$

under the initial condition

$$u(x,y,0) = u_0(x,y) = x + y. \quad (11)$$

Applying the homotopy perturbation method to Equation(10),we have

$$u_t + p[(-u_{xy} - uu_x - u)] = 0. \quad (12)$$

In the view of HPM, we use the homotopy parameter p to expand the solution

$$u(x,y,t) = u_0 + pu_1 + p^2u_2 + ... \tag{13}$$

The approximate solution can be obtained by taking p=1 in equation (13)as

$$u(x,y,t) = u_0 + u_1 + u_2 + ... \tag{14}$$

Now substituing equation(12)into equation(11)and equating the terms with identical powers of p, we obtain the series of linear equations, which can be easily solved.First few linear equations are given as

$$p^0 : \frac{\partial u_0}{\partial t} = 0 \tag{15}$$

$$p^1 : \frac{\partial u_1}{\partial t} = \frac{\partial^2 u_0}{\partial x \partial y} + \frac{\partial u_0}{\partial x}\frac{\partial u_0}{\partial y} + u_0. \tag{16}$$

$$p^2 : \frac{\partial u_2}{\partial t} = \frac{\partial^2 u_1}{\partial x \partial y} + \frac{\partial u_1}{\partial x}\frac{\partial u_0}{\partial y} + \frac{\partial u_0}{\partial x}\frac{\partial u_1}{\partial y} + u_1. \tag{17}$$

Using the initial condition(11), the soution of Equation(15)is given by

$$u(x,y,0) = u_0(x,y) = (x+y). \tag{18}$$

Then the solution of Equation(16)will be

$$u_1(x,y,t) = \int_o^t \left(\frac{\partial^2 u_0}{\partial x \partial y} + \frac{\partial u_0}{\partial x}\frac{\partial u_0}{\partial y} + u_0 \right) dt. \tag{19}$$

$$u_1(x,y,t) = (x+y)t + t. \tag{20}$$

Also, We can find the solution of Equation(17)by using the following formula

$$u_1(x,y,t) = \int_0^t \left(\frac{\partial^2 u_1}{\partial x \partial y} + \frac{\partial u_1}{\partial x}\frac{\partial u_0}{\partial y} + \frac{\partial u_0}{\partial x}\frac{\partial u_1}{\partial y} + u_1 \right) dt. \tag{21}$$

$$u_2(x,y,t) = (x+y)\frac{t^2}{2!} + \frac{3t^2}{2!} \tag{22}$$

etc.therefore, from equation(18),the approximate solution of equation(10)is given as

$$u(x,y,t) = (x+y) + (x+y)t + t + \frac{t^2}{2!} + \frac{3t^2}{2!} + ... \tag{23}$$

Hence the exact solution can be expressed as

$$u(x,y,t) = (x+y)e^t + (n!+1)(e^t - 1), \tag{24}$$

provided that $0 \leq t < 1$

1.2 Adomain Decomposition Method(ADM)

Consider the following linear operatpor and their inverse operators:

$$L_t = \frac{\partial}{\partial t}; L_{x,y} = \frac{\partial^2}{\partial x \partial y}.$$

$$L_t^{-1} = \int_0^t (.)dt, L_{x,y} = \int_0^x \int_0^y (.)d\tau d\gamma.$$

Using the above notations,Equation(1)becomes

$$L_t(u) = L_{x,y}(u) + u_x u_y, \tag{25}$$

Operating the inverse operators L_t^{-1} to equation(25)and using the initial condition gives

$$u(x,y,t) = u_0(x,y,t) + L_t^{-1}(L_{x,y}(u)) + L_t^{-1}(u_x u_y), \tag{26}$$

The decomposition method consists of representing the solution u(x,y,t)by the decomposition series

$$u(x,y,t) = \sum_{q=0}^{\infty} u_q(x,y,t). \tag{27}$$

The nonlinear term $u_x u_y$ is represented by a series of the so called Adomain polynomials, given by

$$u = \sum_{q=0}^{\infty} A_q(x,y,t). \tag{28}$$

The component $u_q(x,y,t)$ of the solution u(x,y,t)is determined in a recursive manner.Replacing the decomposition series (27)and(26)gives

$$\sum_{q=0}^{\infty} u_q(x,y,t) = u_0(x,y,t) + L_t^{-1}(L_{x,y,t}(u)) + L_t^{-1} \sum_{q=0}^{\infty} A_q(x,y,t) \tag{29}$$

According to ADM the zero-th component $u_0(x,y,t)$ is identified from the initial or boundary cndi-tions and from the source terms.The remaining components of $u(x,y,t)$ are determined in a recursion manner as follows

$$u_0(x,y,t) = u_0(x,y), \tag{30}$$

$$u_k(x,y,t) = L_t^{-1}(L_{x,y}(u)) + L_t^{-1}(A_k), k \geq 0, \tag{31}$$

Where thhe adomain polynomials for the nonlinear term $u_x u_y$ are derived from the following recursive formulation

$$A_k = \frac{1}{k!} \frac{d^k}{d\lambda^k} \left(\left(\sum_{i=0}^{\infty} \lambda^i u_i \right) \left(\sum_{i=0}^{\infty} \lambda^i u_i \right) \right)_{\lambda=0}, k = 0,1,2,... \tag{32}$$

First few Adomian polynomials are given as

$$A_0 = u_0 \frac{\partial u_0}{\partial x_1}, A_1 = u_0 \frac{\partial u_1}{\partial x_1} + u_1 \frac{\partial u_0}{\partial x_1},$$

$$A_2 = u_2\frac{\partial u_0}{\partial x_1} + u_1\frac{\partial u_1}{\partial x_1} + u_0\frac{\partial u_2}{\partial x_1}, \tag{33}$$

using equation(31)for the adomain polynomials A_k,we get

$$u_0(x,y,t) = u_0(x,y), \tag{34}$$

$$u_1(x,y,t) = L_t^{-1}(L_{x,y}(u_0)) + L_t^{-1}(A_0), \tag{35}$$

$$u_2(x,y,t) = L_t^{-1}(L_{x,y}(u_1)) + L_t^{-1}(A_1), \tag{36}$$

and so on.Then the q-th term, u_q can be determined from

$$u_q = \sum_{0}^{q-1} u_k(x,y,t). \tag{37}$$

Knowing the components of u, the analytical solution follows immediately.

1.2.1 Computational illustrations of ADM for (n+1)-dimensional Burgers' equation with damping

using Equations(32) and (33), first few components of the decomposition series are given by

$$u_0(x,y,t) = (x+y), \tag{38}$$

$$u_1(x,y,t) = (x+y)t + t, \tag{39}$$

$$u_2(x,y,t) = (x+y)\frac{t^2}{2!} + \frac{3t^2}{2!}, \tag{40}$$

$$u_3(x,y,t) = (x+y)\frac{t^3}{3!} + \frac{7t^3}{3!}, \tag{41}$$

$$u(x,y,t) = \sum_{k=0}^{\infty} u_k(x,y,t)$$

$$= u_0(x,y,t) + u_1(x,y,t) + u_2(x,y,t) + ...,$$

$$= (x+y) + (x+y)t + t + \frac{3t^2}{2!} + (x+y)\frac{t^2}{2!} + (x+y)\frac{t^3}{3!} + \frac{7t^3}{3!} + ... \tag{42}$$

Hence, the exact solution can be expressed as

$$u(x,y,t) = (x+y)e^t + (n!+1)(e^t - 1) \tag{43}$$

1.3 Differential Transform Method(DTM)

In this section,we give some basic definitions of the differential transformation.Let D denotes the differential transform operator and D^{-1} the inverse differential transform operator.

1.3.1 Basic Definition of DTM

Definition:1

If $u\left(x_1, x_2, ..., x_n, t\right)$ is analytic in the domain Ω then its (n+1)- dimensional differential transform is given by

$$\begin{aligned} U\left(k_1, k_2, ..., k_n, k_{n+1}\right) = &\left(\frac{1}{k_1!, k_2!, ..., k_n!, k_{n+1}!}\right) \times \\ &\frac{\partial^{k_1+k_2+...+k_n+k_{n+1}}}{\partial_{x_1}^{k_1} \partial_{x_2}^{k_2} ... \partial_{x_n}^{k_n} \partial_{x_t}^{k_{n+1}}} .u(x_1, x_2, ..., x_n, t)|x_1 = 0, x_2 = 0, ..., x_n = 0, t = 0 \end{aligned} \tag{44}$$

where

$$\begin{aligned} u(x_1, x_{2,...,x_n,t}) &= \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} ... \sum_{k_n=0}^{\infty} \sum_{k_{n+1=0}}^{\infty} U\left(k_1, k_2, ..., k_n, k_{n+1}\right) .x_1^{k_1} x_2^{k_2} ... x_n^{k_n} t^{k_{n+1}} \\ &= D^{-1}\left[U\left(k_1, k_2, ..., k_n, k_{n+1}\right)\right] \end{aligned} \tag{45}$$

Definition:2

If $u(x_1, x_2, ..., x_n, t) = D^{-1}[u(k_1, k_2, ..., k_n, k_{n+1})], v(x_1, x_2, ..., x_n, t) = D^{-1}[V(k_1, k_2, ..., k_n, k_{n+1})]$,and $\otimes$ denotes convolution,then the fundamental operations of the differential transform are expressed as follows:

$$\begin{aligned} (a).D[u(x_1, x_2, ..., x_n, t)v(x_1, x_2, ..., x_n, t)] &= U(k_1, k_2, ..., k_n, k_{n+1}) \otimes V(k_1, k_2, ..., k_n, k_{n+1}) \\ &= \sum_{a_1=0}^{k_1} \sum_{a_2=0}^{k_2} ... \sum_{a_n=0}^{k_n} \sum_{a_{n+1}=0}^{k_{n+1}} U(a_1, k_2 - a_2, ..., k_{n+1} - a_{n+1})V(k_1 - a_1, a_2, ..., a_{n+1}) \end{aligned} \tag{46}$$

$$(b).D[\alpha u(x_1, x_2, ..., x_n, t) \pm \beta v(x_1, x_2, ..., x_n, t)] = \alpha U(k_1, k_2, ..., k_n, k_{n+1}) \tag{47}$$

$$\begin{aligned} (c).D\frac{\partial^{k_1+k_2+...+k_n+k_{n+1}}}{\partial_{x_1}^{k_1} \partial_{x_2}^{k_2} ... \partial_{x_n}^{k_n} \partial_{x_t}^{k_{n+1}}} .u(x_1, x_2, ..., x_n, t) &= (k_1 + 1)(k_1 + 2)...(k_1 + r_1)(k_2 + 1)(k_2 + 2) \\ &...(k_2 + r_2)...(k_{n+1} + 1)(k_{n+1} + 2)...(k_{n+1} + r_{n+1}).U(k_1 + r_1, ..., k_{n+1} + r_{n+1}) \end{aligned} \tag{48}$$

1.3.2 Computational illustrations of (2+1)-dimensional Burgers' equation with damping

Here we describe the method explained in the previous section, by the following to validate the efficiency of the DTM Consider the (2+1)-dimensional Burgers' equation with damping as,

$$u_t = u_{xy} + u_x u_y + u, \tag{49}$$

Subject to the initial condition

$$u(x, y, 0) = u_0(x, y) = x + y \tag{50}$$

Talking the differential transform of equation(49).,we have

$$\begin{aligned} (k_3 + 1)U(k_1, k_2, k_3 + 1) = &(k_1 + 2)(k_1 + 1)U(k_1 + 1, k_2 + 1, k_3) + U(k_1, k_2, k_3) \\ &+U(a_1, k_2 - a_2, k_3 - a_3) + U(k_1 - a_1, a_2, k_3 - a_3) \end{aligned} \tag{51}$$

From the initial condition equation(50),it can beseen that

$$u(x,y,0)=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}U(k_1,k_2,0)(x^{k_1}.y^{k_2})=x+y \tag{52}$$

Where

$$U(k_1,k_2,0)=\begin{cases}1 & \text{if } k_i=1,k_j=0,\ i\neq j\text{,i,j=1,2}\\ 0 & \text{otherwise}\end{cases} \tag{53}$$

Using eq.(53)into eq.(52)one can obtain the values of $U(k_1,k_2,k_3,k_4)$ as follows

$$U(k_1,k_2,1)=\begin{cases}1 & \text{if } k_i=1,k_j=0,\ i\neq j\text{,i,j=1,2}\\ 0 & \text{otherwise}\end{cases} \tag{54}$$

$$U(k_1,k_2,2)=\begin{cases}1 & \text{if } k_i=1,\ k_j=0,i\neq j,\ i,\ j=1,2\\ 0 & \text{otherwise}\end{cases} \tag{55}$$

Then from eqn.(45)we have

$$u(x,y,t)=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}U(k_1,k_2,k_3)x^{k_1}y^{k_2}t^{k_3}$$

$$=(x+y)+(x+y)t+t+\frac{3t^2}{2!}+(x+y)\frac{t^2}{2!}+(x+y)\frac{t^3}{3!}+\frac{7t^3}{3!}+... \tag{56}$$

Hence,the exact solution is given by

$$u(x,y,t)=(x+y)e^t+(n!+1)(e^t-1) \tag{57}$$

1.4 Conclusion

1.In this work,homotopy perturbation methhod,DTM and adomian decomposition method have been successfully applied for solving (2+1)-dimensional Burgers' equation with damping term.

2.The solutions obtained by these methods are an infinite power series for an appropriate initial condition, which can,in turn be expressed in a closed form, the exact solution.

3.The results reveal that the methods are very effective, convenient and quite accurate mathematical tools for solving the (2+1)-dimensional Burgers' equation equation with damping.

4.The solution is calculated in the form of the convergent power series with easily computable components.

5.These method, which can be used without any need to complex computations.

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