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Comparative Analysis of Different Numerical Methods of Solving First Order Differential Equation

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ABSTRACT

A mathematical equation which relates various function with its derivatives is known as a differential equation.. It is a well known and popular field of mathematics because of its vast application area to the real world problems such as mechanical vibrations, theory of heat transfer, electric circuits etc. In this paper, we compare some methods of solving differential equations in numerical analysis with classical method and see the accuracy level of the same. Which will helpful to the readers to understand the importance and usefulness of these methods.

Keywords: *Differential Equations, Runge Kutta methods , Step size*

1. Brief History of Ordinary Differential Equations and Numerical Analysis

Differential equation is the resultant equation when one can remove constants from a given mathematical equation. In point of vies of many historians of mathematics, the study of ordinary differential equations started in 1675. In this year, Gottfried Leibniz(1646-1716)[5] wrote the equation

$$\int x dx = \frac{x^2}{2}$$

First order differential equation defined be Isaac Newton (1646-1727)[7] into three different classes

1. $\frac{dy}{dx} = f(x)$
2. $\frac{dy}{dx} = f(x, y)$
3. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$

Later on lots of work published by Leibniz, Bernoulli[2]. The initial finding of general method of integrating Ordinary differential equation ended by 1775. Then onwards an till today differential equation is an vital part of mathematics.

Earliest mathematical writings on numerical analysis is a Babylonian tablet from the Yale Babylonian collection (YBC 7289), which gives a sexagesimal numerical approximation of $(2)^{1/2}$, the length of the diagonal in a unit square. Modern numerical analysis does not seek exact solutions because exact solutions are often impossible to obtain in practice. So it is working to found approximate answers with same bounds on errors. There are a number of methods to solve first order differential equations in numerical analysis. For comparative analysis we pick some of those methods, remembering the fact that in each method the standard differential equation of first order is as follows :

$y'(t) = f(t, y(t))$ where $y(t_0) = y_0$.

1.1 Euler's Method :

This method is named after Leonhard Euler, who treated in his book *institutionum calculi integralis* volume-III, published in 1978. It is the first order method because it uses straight-line segment for the approximation of the solution. and also it is the simplest Runge kutta method. The formula used in this method is given by the following expression :

$$y(t+h) = y(t) + hf(t, y(t)),$$

Where h is the step size. This formula is also known as *Euler-Cauchy* or the *point-slope* formula[4]. The local and global error is O(h^2) and O(h) respectively.

1.2 Heun's Method :

This method is some time refer to the improved[10] or modified Euler[1] method. This method is named after Karl Heun. It is nothing but the extension of Euler method. In this method, we used two derivatives to obtained an improved estimate of the slope for the entire interval by averaging them. This method is based on two values of the dependent variables the predicted values $\widetilde{y}_{i+1}$ and the final values y_{i+1}, which are given by

$$\widetilde{y}_{i+1} = y_i + hf(t_i, y_i)$$

$$y_{i+1} = y_i + \frac{h}{2}[f(t_i, y_i) + f(t_{i+1}, \widetilde{y}_{i+1})]$$

In this method, one can trying to improve the estimate of the determination of two derivatives for the interval. In this method local and global errors are $O(h^3)$ & $O(h^2)$ respectively.

1.3 Heun's Method of Order Three :

The third order Heun's method formula is as follows :

$$y_{n+1} = y_n + \frac{h}{4}(k_1 + 3k_3)$$

Where $k_1 = f(t_n, y_n)$,

$$k_2 = f\left(t_n + \frac{h}{3}, y_n + \frac{h}{3}k_1\right),$$

$$k_3 = f\left(t_n + \frac{2h}{3}, y_n + \frac{2}{3}hk_2\right).$$

In this method local and global errors are $O(h^4)$ & $O(h^3)$ respectively.

1.4 Fourth Order Runge Kutta Method[3] :

The most popular RK methods are fourth ordr. The following is the most commonly used form and that is why we will call it as classical fourth order RK method.

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Where $k_1 = hf(t_n, y_n)$,

$$k_2 = hf\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right),$$

$$k_3 = hf\left(t_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right).$$

$$k_4 = hf\left(x_n + h, y_n + k_3\right).$$

It is noted that, if the derivative is a function of only x, then this method reduces to *Simpson's 1/3rd rule*. These methods are developed by German mathematicians C. Runge and M.W. Kutta near by 1900.

1.5 Classical Method (Linear Differential Equations of first order) :

As we discussed above, the existence of differential equations[9] was discovered with the invension of calculus by Leibniz and Newton. A linear differential equation[8] is of the following type

$$\frac{dy}{dx} + Py = Q \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 1.5.1$$

where P and Q are functions of x. The general solution of the equation 1.5.1 is

$$y\,(\text{I.F.}) = \int Q\,(\text{I.F.})dx + \text{c} \ldots\ldots\ldots\ldots 1.5.2$$

where c is the arbitrary constant and eliminated by the given initial condition and

I.F. = Integrating factor = $e^{\int Pdx}$.

2. Comparison and Analysis :

We consider the following well known differential equation for the comparison for methods mentioned in this paper:

$$\frac{dy}{dx} = x + y \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 2.1$$

subject to the condition $y(0) = 1$, which is clearly a linear differential equation. When we compare it with the standard linear differential equation of first order, then we get the integrating factor of this equation as e^{-x} and the general solution is as follows :

$$y = ce^x - x - 1$$

And by the given initial condition, we have

$$y = 2e^x - x - 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots 2.2$$

which is the required particular solution of the equation 2.1. With the help of solution 2.2 of the differential equation 2.1 we get different y corresponding to different x. We restricted our self for $x = 4$. By this value of x and from equation 2.2 we get the exact value of y as 104.1963001.

Now, we apply all above four methods one by one for finding the Numerical solution of equation 2.1. We consider the step size for every method as 0.4. Table 2.1 shows that the values of y in between $x = 0$ to $x = 4$.

Table 2.1

Sr. No.	x	Euler's Method	Heun's Method	Heun's Method of order three	RK Method of order Four	Exact solution by Linear DE Method
		$f(x)$				
1	0	1	1	1	1	1
2	0.4	1.4	1.56000000	1.58133333	1.58346667	1.58364939
3	0.8	2.12	2.58080000	2.64417422	2.65053668	2.65108186
4	1.2	3.288	4.28358400	4.42478237	4.43901391	4.44023384
5	1.6	5.0832	6.99570432	7.27534226	7.30363835	7.30606485
6	2	7.75648	11.20164239	11.72084353	11.77358745	11.77811220
7	2.4	11.659072	17.61843074	18.54387075	18.63825285	18.64635276
8	2.8	17.2827008	27.30727750	28.91099667	29.07519638	29.08929354
9	3.2	25.31578112	41.83877070	44.56119236	44.84102629	44.86506039
10	3.6	36.722093568	63.53738063	68.08668408	68.55613361	68.59646889
11	4	52.8509309952	95.84332334	103.35161707	104.12944305	104.1963001

Now after finding the values of y from different numerical method , we compare the every value of y of exact solution(last Column of table 2.1) with corresponding approximate value in each method with the help of following line figures.

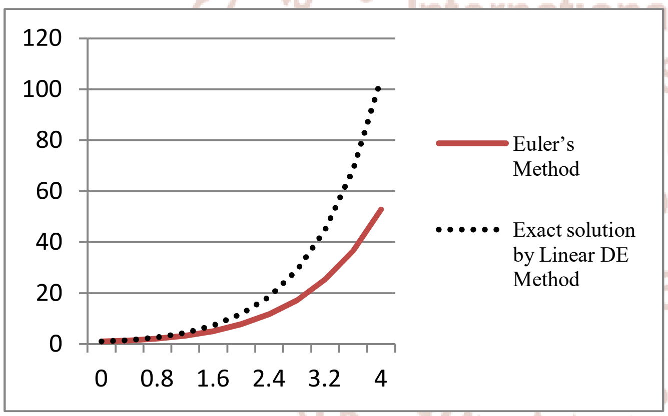

Figure – 2.1

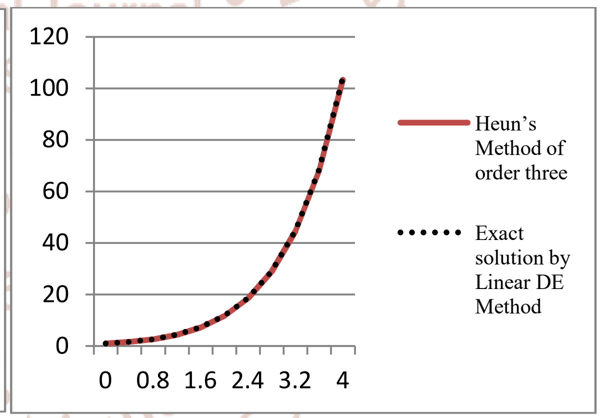

Figure -2.3

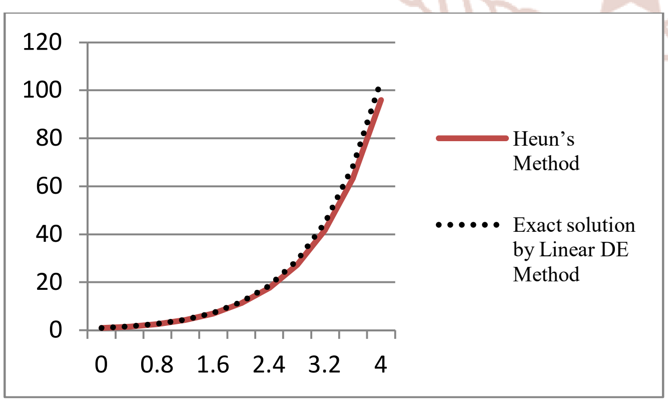

Figure – 2.2

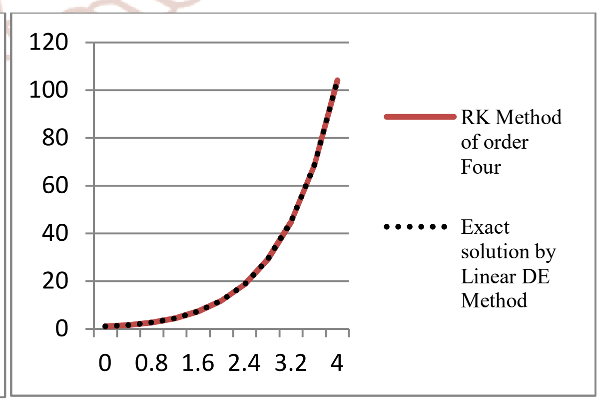

Figure -2.5

Figure -2.1 shows the comparison between Euler's method and exact solution by Linear Differential equation method, similarly Figure -2.2, Figure -2.3 and Figure -2.4 shows the comparison between Heun's method, of order two, Heun's method of order three & Runge-Kutta method of order four with exact solution by Linear Differential equation method respectively. We can see that Runge Kutta method is nearest to the exact solution which is 104.1963. Euler's method shows the maximum variation when step size is 0.4.

In the next level we compare the value of y when $x = 4$ with different step size with the help of table no. 2.2.

Table 2.2

Sr. No.	Step Size	Euler's Method	Heun's Method	Heun's Method of order three	RK Method of order Four
1	0.4	52.8509	95.8433	103.3516	104.1294
2	0.2	71.6752	101.7153	104.0722	104.1914
3	0.1	85.5185	103.5228	104.1795	104.1960
4	0.05	94.1229	104.0211	104.1941	**104.1963**
5	0.04	96.0099	104.0833	104.1952	**104.1963**

The above table shows that the accuracy level of each method is increases if we increase the step size. That is if we divide the interval (0, 4) into 100 equal parts then we will get the more accurate results. As we know that the Runge Kutta method is the most accurate method up to four places of decimals, here when we take step size 0.04 then we reached up to that level of accuracy.

4. Conclusion : In this paper, we showed with the help of a linear differential equation of first order that the solution by classical method of solving differential equations is almost equal with the solutions obtained by numerical methods. When we took 100 partitions, the Runge Kutta method of fourth order give the most accurate answer up to four places of decimals. For this much accuracy in Heun's method of order three, we require step size .026app. i.e. 150partitions, for the same accuracy level in the Heun's method of order two, we required step size 0.02 i.e. 200 partitions and for Euler method we required more than 1000 partitions of the given interval. This is very useful information, with the help of this example, to understand the importance of these methods specially Runge Kutta method of fourth order. The thing which we have to remember that to reach up to the highest accuracy level we have to reduce the step size as possible as required/possible. If number of partitions of the given or require interval are more than we get the similar results in comparison of non numerical methods of the same**.**

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