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Singular Third-Order Multipoint Boundary Value Problem at Resonance

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ABSTRACT

The present paper is particularly exhibits about the derive results of a third-order singular multipoint boundary value problem at resonance using coindence degree arguments.

Keywords: *The present paper is particularly exhibits about the derive results of a third-order singular multipoint boundary value problem at resonance using coindence degree arguments.*

INTRODUCTION

This paper derive the existence for the third-order singular multipoint boundary value problem at resonance of the form

$$u''' = g(t), u(t), u'(t), u''(t)) + h(t)$$

$$u'(0) = 0, u''(0) = 0,$$

$$u(1) = \bigvee_{i,j=1}^{m-3} a_i b_j \, u(\varsigma_{ij}),$$

Where $g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is caratheodory's function (i.e., for each $(u, v) \in \mathbb{R}^2$ the function $g(., u, v)$ is measurable on [0,1]; for almost everywhere $t \in [0,1]$, the function $g(t,.,.)$ is continuous on $\mathbb{R}^2$). Let $\varsigma_{ij} \in (0,1), i, j = 1,2, \dots, m-3$, and $\bigvee_{i,j=1}^{m-3} a_i b_j = 1$, where g and h have singularity at $t=1$.

In [1] Gupta et al. studied the above equation when g and h have no singularity and $\bigvee_{i,j=1}^{m-3} a_i b_j \neq 1$. They obtained existence of a $C^1[0,1]$ solution by utilizing Letay-Schauder continuation principle. These results correspond to the nonresonance case. The scope of this article is therefore to obtained the survive results when $\bigvee_{i,j=1}^{m-3} a_i b_j = 1$ (the resonance case) and when g and h have a singularity at $t = 1$.

Definition 1

Let U and W be real Banach spaces. One says that the linear operator $L: dom\, L \subset U \to W$ is a **Fredholm mapping of index zero** if Ker L and $W/Im\, L$ are of finite dimension, where $Im\, L$ denotes the image of L.

Note

We will require the continuous projections $P: U \to U$, $Q: W \to W$ such that Im P = Ker L,Ker Q = Im L, U =Ker L$\oplus$ Ker P, W =Im L$\oplus$ Im Q, $L|_{dom\, L\, \cap Ker\, P}$: dom L $\cap$ ker P $\to$ Im L Iis an isomorphism.

Definition 2

Let L be a Fredholm mapping of index zero and Ω a bounded open subset of U such that dom $L \cap \Omega \neq \phi$. The map M: $U \to W$ is called ***L*-compact on $\bar{\Omega}$**, if the map $QN(\bar{\Omega})$ is bounded and $R_P(I - Q)$ is compact, where one denotes by R_P : Im L $\to$ $dom\, L\, \cap Ker\, P$ the generalized inverse of L. In addition M is L-completely continuous if it L-compact on every bounded $\Omega \subset U$.

Theorem 1

Let L be a Fredholm operator of index zero and let N be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied :

(i) $Lu \neq \kappa Mu$ for every $(u,\kappa) \in [(dom\ L \setminus Ker\ L) \cap \partial\kappa] \times (0,1)$.
(ii) $Mu \notin Im\ L$, for every $u \in Ker\ L \cap \partial\kappa$.
(iii) $\deg\left(QM|_{Ker\ L\cap\partial\kappa},\ \kappa \cap Ker\ L, 0\right) \neq 0$,

with $Q: W \to W$ being a continuous projection such that $Ker\ Q = Im\ L$. then the equation $Lu = Mu$ has at least one solution in $dom\ L\ \cap \overline{\Omega}$.

Proof :

We shall make use of the following classical spaces, $C[0,1], C^1[0,1], C^2[0,1], L^1[0,1], L^2[0,1]$,

and $L^\infty[0,1]$. Let $AC[0,1]$ denote the space of all absolute continuous functions on $[0,1]$, $AC^1[0,1] = \{u \in\ C^2[0,1] : u''(t) \in\ AC[0,1]\}$, $L^1_{loc}[0,1] = \left\{u: u|_{[0,d]} \in\ L^1[0,1]\right\}$ for every compact interval $[0,d] \subseteq [0,1]$.

$$AC_{loc}[0,1) = \left\{u: u|_{[0,d]} \in\ AC[0,1]\right\}.$$

Let U be the Banach space defined by

$$U = \{u \in L^1_{loc}[0,1] : (1-t^2)u(t) \in L^1[0,1]\},$$

With the norm

$$\|v\|_u = \int_0^1 (1-t^2)\,|v(t)|dt.$$

Let U be the Banach space

$U = \{u \in C^2[0,1): u \in C[0,1], \lim_{t\to 1^-}(1-t^2)\,u''\ exists\}$,

With the norm

$$\|u\|_u = \max\left\{\ \|u\|_\infty, \left\|(1-t^2)u''(t)\right\|_\infty\right\}. \quad (1)$$

Where $\|u\|_\infty = \sup_{t\in[0,1]}|u(t)|$.

We denote the norm in $L^1[0,1]$ by $\|.\|_1$. we define the linear operator $L: dom\ L \subset U \to W$ by

$$Lu = u'''(t), \quad (2)$$

Where

$$dom\ L = \left\{u \in U :\ u'(0) = 0, u''(0) = 0, u(1) = \bigvee_{i,j=1}^{m-3} a_i b_j u(\varsigma)\right\}$$

And $M: U \to W$ is defined by

$$Lu = g\ \left(t, u(t), u'(t), u''(t)\right) + h(t). \quad (3)$$

Then boundary value problem (1) can be written as

$Lu = Nu.$

Therefore $L = N.$

Lemma

If $\bigvee_{i,j=1}^{m-3} a_i b_j = 1$ then

(i) $Ker\ L = \{u \in dom\ L: u(t) = c, c \in \mathbb{R}, t \in [0,1]\}$;
(ii) $Im\ L = \left\{\begin{matrix} v \in z: \\ \bigvee_{i,j=1}^{m-3} a_i b_j \int_{\zeta_i}^1 \int_{\zeta_j}^1 \int_0^s v(\varrho)d\varrho ds = 0\end{matrix}\right\}$
(iii) $L: dom\ L\ \subset U \to W$ is a Fredholm operator $Q: W \to W$ can be defined by

$$Qv = \frac{e^t}{h} \bigvee_{i,j=1}^{m-3} a_i b_j \int_{\zeta_i}^1 \int_{\zeta_j}^1 \int_0^s v(\varrho)d\varrho ds, \quad (4)$$

Where

$$h = \bigvee_{i,j=1}^{m-3} a_i b_j \left[e + \zeta_i + \zeta_j - e^{\zeta_i} - e^{\zeta_j} - 1\right]$$

(iv) The linear operator $R_p: Im\ L: \to dom\ L\ \cap Ker\ P$ can be defined as

$$R_p = \int_{\zeta_i}^1 \int_{\zeta_j}^1 \int_0^s v(\varrho)d\varrho \quad (5)$$

(v) $\left\|R_p v\right\|_U \leq \|v\|_W$ for all $v \in W$.

Proof :

(i) It is obvious that $Ker\ L = \{u \in dom\ L: u(t) = c, c \in \mathbb{R}\}$.

(ii) We show that

$$Im\, L = \left\{ \substack{v \in W: \\ \bigvee_{i,j=1}^{m-3} a_i b_j \int_{\zeta_i}^1 \int_{\zeta_j}^1 \int_0^s v(\varrho) d\varrho ds = 0} \right\}. \quad (7)$$

To do this, we consider the problem

$$w'''(t) = v(t) \quad (8)$$

And we show that (5) has a solution $w(t)$ satisfying

$$w''(0) = 0,\ w'(0) = 0, w(t) = \bigvee_{i,j=1}^{m-3} a_i b_j w(\zeta_i \zeta_j)$$

If and only if

$$\bigvee_{i,j=1}^{m-3} a_i b_j \int_{\zeta_i}^1 \int_{\zeta_j}^1 \int_0^s v(\varrho) d\varrho ds = 0 \quad (9)$$

Suppose (3) has a solmution $w(t)$ satisfying

$$w''(0) = 0,\ w'(0) = 0, w(t) = \bigvee_{i,j=1}^{m-3} a_i b_j w(\zeta_i \zeta_j)$$

Then we obtain from (5) that

$$w(t) = w(0) + \int_{\zeta_i}^1 \int_{\zeta_j}^1 \int_0^s v(\varrho) d\varrho ds, \quad (10)$$

And applying the boundary conditions we get

$$\bigvee_{i,j=1}^{m-3} a_i b_j \int_{\zeta_i}^1 \int_{\zeta_j}^1 \int_0^s v(\varrho) d\varrho ds = \int_{\zeta_i}^1 \int_{\zeta_j}^1 \int_0^s v(\varrho) d\varrho ds, \quad (11)$$

Since $\bigvee_{i,j=1}^{m-3} a_i b_j = 1$, and using (i) and we get

$$\bigvee_{i,j=1}^{m-3} a_i b_j \int_{\zeta_i}^1 \int_{\zeta_j}^1 \int_0^s v(\varrho) d\varrho ds = 0$$

On the other hand if (6) holds, let $u_0 \in \mathbb{R}$; then

$$w(t) = w(0) + \int_{\zeta_i}^1 \int_{\zeta_j}^1 \int_0^s v(\varrho) d\varrho ds,$$

Where $v \in Z$

$$w'''(t) = v(t) \quad (12)$$

(iii) For $v \in Z$, we define the projection Qv as

$$Qv = \frac{e^t}{h} \bigvee_{i,j=1}^{m-3} a_i b_j \int_{\zeta_i}^1 \int_{\zeta_j}^1 \int_0^s v(\varrho) d\varrho ds,$$

$t \in [0,1]$, (13)

Where

$$h = \bigvee_{i,j=1}^{m-3} a_i b_j \left[e + \zeta_i + \zeta_j - e^{\zeta_i} - e^{\zeta_j} - 1\right] \neq 0.$$

We show that $Q: W \to W$ is well defined and bounded.

$$|Qv(t)| \le \frac{|e^t|}{|h|} \bigvee_{i,j=1}^{m-3} |a_i||b_j| \int_0^1 (1-s)^2 |v(s)| ds$$

$$= \frac{1}{|h|} \bigvee_{i,j=1}^{m-3} |a_i||b_j| \, \|v\|_W |e^t|$$

$$\|Qv\|_W \le \int_0^1 (1-t)^2 |Qv(t)| dt$$

$$\le \frac{1}{|h|} \bigvee_{i,j=1}^{m-3} |a_i||b_j| \, \|v\|_W |e^t| \int_0^1 (1-s)^2 ds$$

$$= \frac{1}{|h|} \bigvee_{i,j=1}^{m-3} |a_i||b_j| \, \|v\|_W \|e^t\|_W .$$

In addition it is easily verified that

$$Q^2 v = Qv, v \in W. \quad (14)$$

We therefore conclude that $Q: W \to W$ is a projection. If $v \in Im\, L$, then from (6) $Qv(t) = 0$. Hence $Im\, L \subseteq Ker\, Q$. Let $v_1 = v - Qv$; that is, $v_1 \in Ker\, Q$. Then

$$\bigvee_{i,j=1}^{m-3} a_i b_j \int_{\zeta_i}^{1}\int_{\zeta_j}^{1}\int_{0}^{s} v_1(\varrho)d\varrho ds$$
$$= \bigvee_{i,j=1}^{m-3} a_i b_j \int_{\zeta_i}^{1}\int_{\zeta_j}^{1}\int_{0}^{s} v(\varrho)d\varrho ds$$
$$-\frac{1}{h}\bigvee_{i,j=1}^{m-3} \int_{\zeta_i}^{1}\int_{\zeta_j}^{1}\int_{0}^{s} v(\varrho)d\varrho ds$$

Here $h = 0$ we get

$$\bigvee_{i,j=1}^{m-3} a_i b_j \int_{\zeta_i}^{1}\int_{\zeta_j}^{1}\int_{0}^{s} v_1(\varrho)d\varrho ds$$
$$= \bigvee_{i,j=1}^{m-3} a_i b_j \int_{\zeta_i}^{1}\int_{\zeta_j}^{1}\int_{0}^{s} v(\varrho)d\varrho ds$$

Therefore $v_1 = v$.

Thus $v_1 \in Im\, L$ and therefore $Ker\, Q \subseteq Im\, L$ and hence

$$W = Im\, L + Im\, Q = Im\, L + \mathbb{R}.$$

It follows that since $Im\, L \cap \mathbb{R} = \{0\}$, then $W = Im\, L \oplus Im\, Q$.

Therefore,

$\dim Ker\, L = \dim Im\, Q = \dim \mathbb{R} = codim\, Im\, L = 1.$

This implies that L is Fredholm mapping of index zero.

(iv) We define $P : W \to W$ by

$$Pu = u(0), \qquad (15)$$

And clearly P is continuous and linear and $P^2 u = P(Pu) = Pu(0) = u(0) = Pu$ and $Ker\, P = \{u \in U : u(0) = 0\}$. We now show that the generalized inverse $K_P = Im\, L \to dom\, L \cap Ker\, P$ of L is given by

$$R_p v = \int_{\zeta_i}^{1}\int_{\zeta_j}^{1}\int_{0}^{s} v(\varrho)d\varrho \qquad (16)$$

For $v \in Im\, L$ we have

$$(LR_P)v(t) = \left[\left(R_p v\right)(t)\right]'' = v(t) \qquad (17)$$

And for $u \in dom\, L \cap Ker\, P$ we know that

$$(R_P\, L)\, u(t) = \int_{\zeta_i}^{1}\int_{\zeta_j}^{1}\int_{0}^{s} u''(\varrho)d\varrho ds$$
$$= \int_0^t (t-s)u''\, ds \qquad (18)$$
$$= u(t) - u'(0)\, t - u(0) = u(t)$$

Since $u \in dom\, L \cap Ker\, P, u(0) = 0$, and $Pu = 0$.

This shows that $R_P = (L|_{dom\, L \cap Ker\, P})^{-1}$.

(v) $\|R_p v\|_\infty \le \max_{t\in[0,1]} \int_0^t (t-s)^2 |v(s)| ds$
$$\le \int_0^t (t-s)^2 |v(s)| ds$$
$$\le \|v\|_{W.}$$

We conclude that

$$\|R_p v\|_W \le \|v\|_{W.}$$

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