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A Study on Orbital Mechanics

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ABSTRACT

Space craft to solar system bodies must deal with multiple deviations under the in uence of gravity. The motion of satellite in one of the types of conic section strongly depends upon escape velocity. The goal of this paper is to nd the trajectory of satellite by analysing orbit's eccentricity and or-bital parameters. This paper also focuses on the derivation of general orbit equation from the base called Newton's law of motion and Newton's law of gravitation.

Keywords: *Eccentricity, Flight path angle, True anomaly, Perigee and Apogee radius, Trajectory.*

1. INTRODUCTION

Mechanics is a branch of science concerned with the relationship between force, displacements, energy and their e ects on physical bodies on their environment. The classical mechanics is one of the sub- elds of mechanics. Classical mechanics is concerned with the some physical laws describing the movement of bodies under the in uence of forces.

2. ORBITAL MECHANICS

Orbital mechanics is also known as astrodynamics. It is the application of celestial mechanics concerning the motion of rockets and other spacecraft. It is concerned with motion of bodies under the in uence of gravity, including both spacecraft and natural astronomical bodies such as star, planets, moons and comets.

2.1 TRAJECTORIES:

Three general types of paths are possible under the gravitational in uence.

An elliptical orbit is a orbit with an eccentricity of less than 1. This includes the special case called circular orbit and its eccentricity is equal to 0.

This paper examines the classifications, cause; types and the possible solutions of conflict in organization. This discovered that conflict generates considerable ambivalence and leave many practitioners and scholars quite uncertain about how best to cope with it. Conflicts are inevitable in human life and also inevitable in organizations even between countries. Conflict occurs in organizations as a result of competition for supremacy, scarcity of economic resource and leadership style. The study also revealed that conflict in organization could be constructive or destructive which can lead to low production or good solution to production.

A hyperbolic trajectory is the trajectory of any body around a cen-tral mass with more enough speed to escape the central object's grav-itational pull. The orbital eccentricity will be greater than one

A parabolic trajectory is an orbit whose eccentricity will be equal to 1. A body travelling along an escape orbit will coast along a parabolic trajectory to in nity and never return.

2.1.1 ESCAPE VELOCITY:

Escape velocity is the velocity required at a given position to establish a parabolic path. Velocities greater than escape velocity result in hyperbolic orbits. Lower velocities result in closed elliptical orbits.

2.2 ORBITAL PARAMETERS:

Apogee (q) refers to the farthest distance between satellite and the Earth.

Perigee (p) refers to the closest distance between satellite and the Earth.

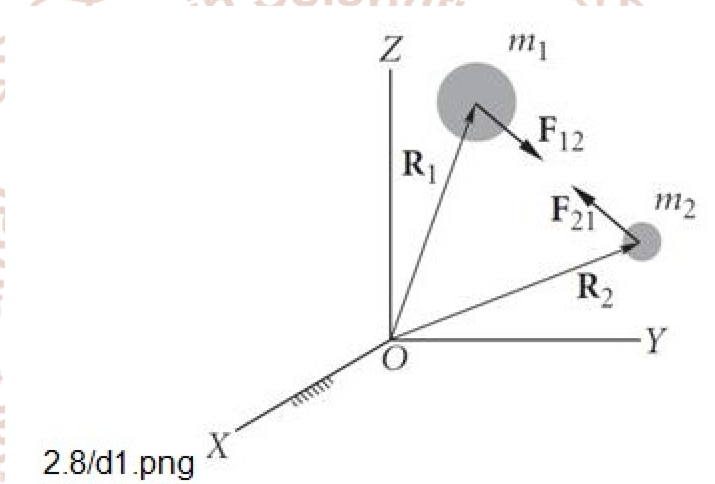
Semi-Major Axis (a): The distance from the center of the orbit ellipse to satellite's apogee or perigee point.

Semi-Minor Axis (b): The shortest distance from the true center of the orbit ellipse to the orbit path.

Period (T): The time required for the satellite to orbit the Earth once.

Pitch angle or ight path angle: The angle between the longitudi-nal axis (where the airplane is pointed) and the horizon.

3. ORBIT EQUATION



Two objects are located in inertial frame of reference XYZ. Each of the bod-ies is acted upon by the gravitational attraction of the other. F_{12} is the force exerted on m_1 by m_2, and F_{21} is the force exerted on m_2 by m_1. The position vector R_G of the center of mass G is de ned by,

$$R_G = \frac{m_1 R_1 + m_2 R_2}{m1+m2} \quad ! (1)$$

Absolute velocity, $v_G = R0_G = \frac{m1R10 + m2R20}{m1+m2}$! (2)

Absolute acceleration, $a = R^{00} = \frac{m_1 R_1^{00} + m_2 R_2^{00}}{m_1+m_2}$! (3)

The term absolute means that the quantities are measured relative to an inertial frame of reference.

Let r be position vector of m_2 relative to m_1.

$$r = R_2 \quad R_1 \quad ! (4)$$

Let uc_r be the unit vector pointing from m_1 towards m_2,

$$u = ^{r} \quad ! (5)$$

c_r r

where r = krk is the magnitude of r.

The force exerted on m_2 by m_1 is,

$$F_{21} = \frac{Gm_1m_2(\ u_{b_r})}{r^2} = \frac{Gm_1m_2(u_{b_r})}{r^2} \quad ! \ (6)$$

where uc_r shows that the force vector F_{21} is directed from m_2 towards m_1.

Applying Newton's second law of motion on body m_2 is $F_{21} = m_2R_2^{00}$, where R_2^{00} is the absolute acceleration of m_2.

$$\frac{Gm_1m_2(u_{b_r})}{r^2_2} = m_2R^{00} \quad ! \ (7)$$

By Newton's third law of motion, $F_{12} = F_{21}$, so for m_1 we have,

$$\frac{Gm_1m_2(u_{b_r})}{r^2} = m_1R_1^{00} \quad ! (8)$$

Equations (7)and (8) are the equations of motion of two bodies in inertial space. By adding ,we get $m_1R_1^{00} + m_2R_2^{00} = 0$! (9).

According to Equation(3) , the acceleration of the center of mass G of the system of two bodies m_1 and m_2 is zero.

Multiplying Equation (7) by m_1 and Equation (8) by m_2, we get

$$\frac{Gm_1^2m_2\ (u_r)}{} = m_1 m_2 R \quad ! \quad (10)$$

$$\frac{Gm_1\ m\ b}{r^2} = m_1\ m_2\ R_1^{00} \ ! \ (11)$$

Subtracting the equation(11) from (10) yields

$$m_1m_2(R_1^{00}\ R_2^{00}) = \frac{Gm_1m_2}{r^2}(m_1 + m_2)u_{c_r} \ ! \ (12)$$

factor m_1m_2

$$r'' = \frac{G(m_1+m_2)}{r^2} u_{c_r} \ ! \ (13)$$

Let the gravitational parameter be de ned as

= G(m_1 + m_2) ! (14)

Using Equation (14) and Equation (5), we can write Equation (13) as

$r'' = \ _{r^3} r$! (15)

This is the second order di erential equation that governs the motion of m_2 relative to m_1.

Let r_1 and r_2 be the position vectors of m_1 and m_2 relative to the center of mass G. The equation of motion of m_2 relative to the center of mass is

$$\frac{Gm_1m_2(u_{br})}{r^2} = m_2 r^{00}_2 \ !(16)$$

where r is the position vector of m_2 relative to m_1. In terms of r_1 and r_2,

$r = r_2 \quad r_1$

Since the position vector of the center of mass relative to itself is zero, we get,

$m_1r_1 + m_2r_2 = 0$

$r_1 = \frac{m_2}{m_1} r_2$

$r = \frac{m_1+m_2}{m_1} r_2$

Substituting this back into Equation (16) and using $(uc_r) = r_2 = r_2$, we get

$$\frac{Gm^3_1 m_2}{(m_1+m_2)^2 r_2^3} r_2 = m_2 r^{00}_2$$

Simplifying,

$$\left(\frac{m_1}{m_1+m_2}\right)^3 \frac{r_2}{r_2^3} = r^{00}_2 \ ! \quad (17)$$

let $\mu' = \left(\frac{m_1}{m_1+m_2}\right)^3$

$$\frac{\mu'}{r_2^3} r_2 = r^{00}_2$$

Similarly, the equation of motion of m_1 relative to the center of mass is found to be

$$\frac{\mu''}{r_1^3} r_1 = r^{00}_1$$

where $\mu'' = \left(\frac{m_2}{m_1+m_2}\right)^3$

The angular momentum of body m_2 relative to m_1 is the moment of m_2's relative linear momentum m_2r^0

$$H_{2=1} = r \times m_2 r^0$$

Divide this equation by $m_2$and let $h = H_{2=1}=m_2$, so that

$$h = r \times r^0 \ ! \ (18)$$

h is the relative angular momentum of m_2 per unit mass, that is, the speci c relative angular momentum. Taking the time derivative of h yields

$\frac{d}{dt}h = r^0 \times r^0 + r\ r^{00}$

But $r^0 \times r^0 = 0$. According to Equation (15) r" = $_{r3}$ r, we have

$r \times r^{00} = r \times (\,_{r3}\,\overline{r}) = \,_{r3}\,(r \overline{\times} r) = 0\ \frac{d}{dt}h$

$\overline{=}0$! (19)

the unit vector,

$hb = \frac{h}{h}$! (20)

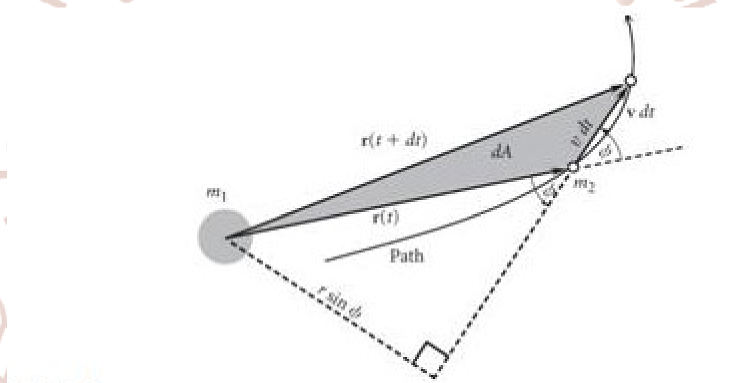
Let us divide the relative velocity vector r^0 into components $V_r = v_r ub_r$ and $V_p = v_p ub_p$, where ub_r and

ub_p are radial and perpendicular unit vector

$$h = rub_r \times (v_r ub_r + v_p ub_p) = rv_p h^b\ \ h = rv_p \ ! (21)$$

The angular momentum depends only on the perpendicular component of the relative velocity.

During the di erential time interval dt the position vector r sweeps out an area dA.
From the gure the triangular area dA is given by



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$$dA = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times vdt \times r \sin\ \ = \frac{1}{2} r(v\sin\ \ dt) = 1_{2rv_pdt}$$

Therefore, using Equation (21) we get

$$\frac{dA}{dt} = \frac{h}{2} \qquad ! (22)$$

The vector identity also known as the bac - cab rule:

$$A\ \ A \times (B \times C) = B(A:C)\ C(A:B)\ \ ! (23)\ r:r = r^2 \qquad ! (24)$$

$\frac{d}{dt}(r:r) = 2r\frac{dr}{dt}$

But

$\frac{d}{dt}(r:r) = r\frac{d}{dt}r + \frac{d}{dt}r\ r = 2r\frac{dr}{dt}$

Thus, we obtain the identity

$$r{:}r^0 = r{:}r^0 \qquad ! \ (25a)$$

Since r^0 = v and r = krk , this can be written alternatively as

$$r{:}v = krk \frac{kdr}{dt} k \qquad ! \ (25b)$$

Let us take the cross product on both sides of Equation (15) r" = $_{r3}$ r with the speci c angular momentum h:

$$r'' \times h = \ _{r3}\, r \times h \qquad ! \ (26)$$

Since $\frac{d}{dt}(r^0 \times h) = r^{00} \times h + r^0 \times h^0$, the left-hand side can be written

$$r^{00} \times h = \frac{d}{dt}(r^0 \times h) - r^0 \times h^0 \qquad ! \ (27)$$

But according to Equation (19), the angular momentum is constant (h^0 =0), so

$$r^{00} \times h = \frac{d}{dt}(r^0 \times h)$$

The right-hand side of Equation (26) can be transformed into:

$$\frac{1}{r^3} r \times h = \frac{1}{r^3}(r \times (r \times r^0))$$

$$= \frac{1}{r^3}[r(r{:}r^0) \quad r^0(r{:}r)]$$

$$= \frac{1}{r^3}[r(rr^0) \quad r^0(r^2)]$$

$$\frac{(rr^0 \qquad r^0 r)}{} = \ _r 2$$

But

$$\frac{d}{dt}\left(\frac{r}{r}\right) = \frac{(r^0 r \ rr^0)}{r^2} = \left(\frac{rr^0 \ r^0 r}{r^2}\right)$$

Therefore

$$\frac{1}{r^3} r \times h = \frac{d}{dt}\left(\frac{r}{r}\right) \ ! \ (28)$$

Substituting Equations (27) and (28) into Equation (26), we get

$$\frac{d}{dt}(r^0 \times h) = \frac{d}{dt}\left(\frac{r}{r}\right)$$

$$\frac{d}{dt}\left(r^0 \times h - \frac{r}{r}\right) = 0$$

Taking the dot product of h on both sides of Equation (29) yields

$$(r^0 x h):h \qquad (\frac{r.h}{r}) = C . h$$

Since r' x h is perpendicular to both r' and h, so that (r' x h) . h= 0. Likewise, since h = r x r' is perpendicular to both r and r', it follows that r . h = 0. Therefore, we have C . h= 0, i.e., C is perpendicular to h, which is normal to the orbital plane.

Rearranging Equation (29)

$$\frac{r}{r} + e = \frac{1}{} (r^0 x\ h) \quad ! (30)$$

where e = C= . The vector e is called the eccentricity. The line de ned by the e is commonly called the apse line. o obtain a scalar equation, let us take the dot product of both sides of Equation (30) with r

$$\frac{r.r}{r} + e = \frac{1}{} (r:(r^0 x\ h\ )) \quad ! (31)$$

we know the identity

$$A:(BxC) = (AxB):C$$

to obtain

$$r:(r^0 xh) = (rxr^0):h = h:h = h^2 \quad ! (32)$$

Substituting this expression into the right-hand side of Equation (31), and substituting r . r= r^2 on the left yields

$$r + r:e = \frac{h_2}{} \quad ! (33)$$

From the de nition of dot product we have

$$r:e = re\ cos$$

In terms of the eccentricity and the true anomaly, we may write Equation (33) as

$$r + re\ cos\ = \frac{h_2}{}$$

$$r = \frac{h^2}{} \frac{1}{1+e\ cos}$$

This is the orbit equation, and it de nes the path of the body m_2 around m_1, relative to m_1.

3.1 Flight path angle:

The ight path angle is the angle that the velocity vector v = r' makes with the normal to the position vector.

$$tan\ = \frac{v_r}{v_p}$$

$$\tan = \frac{e \sin}{1+e \cos}$$

3.2 Period of orbit (ellipse):

At periapsis, $= 0^o$

$$r_p = \frac{h^2}{} \frac{1}{1+e} \; ! \, (34)$$

At apoapsis, $= 180^o$

$$r_a = \frac{h^2}{} \frac{1}{1 \; e} \; ! \, (35)$$

Let 2a be the distance between from periapsis P to apoapsis A, $2a = r_p + r_a$

$$A, 2a = r_p + r_a$$

$$2a = \frac{h^2}{} \frac{1}{1+e} + \frac{h^2}{} \frac{1}{1 \; e}$$

$$a = \frac{h^2}{} \frac{1}{1 \; e^2}$$

$$b = a \sqrt{1 \; e^2}$$

Area of ellipse = A = ab

By Kepler's second law,

$$\frac{dA}{dt} = \frac{h}{2}$$

$$A = \frac{h}{2} t$$

where A = ab and t = T

$$ab = \frac{h}{2} (T)$$

$$T = \frac{2 \; ab}{h}$$

Substituting the values of a and b,

$$T = \frac{2}{2} \left(\frac{h}{\sqrt{1 \; e^2}} \right)^3$$

From the equation of semi major axis, we have $h = \sqrt{a(1 \; e^2)}$

$$T = \frac{2}{\sqrt{}} a^{3=2}$$

3.3 Eccentricity:

From equations (34) and (35) we have,

$$\frac{r_p}{r_a} = \frac{1\ e}{1+e}$$

$$e = \frac{r_a\ r_p}{r_a + r_p}$$

3.4 NUMERICAL ILLUSTRATIONS:

1. A satellite orbits the earth with a perigee radius of 7000 km and apogee radius of 70000 km. Calculate eccentricity and period of orbit.

Solution:

$$e = \frac{r_a\ r_p}{r_a + r_p}$$

$$= \frac{70000\ 7000}{70000+7000}$$

$e = 0:8182$

Since e 1, it will be an ellipse.

we know that $r = \frac{h^2}{\ }\frac{1}{1+e\cos}$

$r_p = \frac{h^2}{\ }\frac{1}{1+e}$

$7000 = \frac{h^2}{398600}\frac{1}{1+0:8182}$

$h = 71226\ km^2/s.$

Period of orbit, $T = \frac{2}{\ 2}\left(\frac{p\ h}{1\ \ e2}\right)^3$

$$T = \frac{2}{398600^2}\left(\frac{p\ 71226}{1\ \ 0:8182^2}\right)^3$$

T = 7511 seconds

T = 20:89 hours

2. At two points on a geocentric orbit the altitude and anomaly are Z_1 = 1700 km , $_1$ = 130^o and Z_2 = 500 km , $_2$ = 50^o. Find

a) eccentricity

b) angular momentum

c) perigee radius

d) perigee altitude
e) semi major axis

Solution:

$r_1 = R_E + z_1 = 6378 + 1700 = 8078$! (1)

$r_2 = R_E + z_2 = 6378 + 500 = 6878$! (2)

we know that $r = \frac{h^2}{} \frac{1}{1+e \cos}$

(1)) $8078 = \frac{h^2}{(398600)} \frac{1}{1+e \cos 130^o}$

$h^2 = 3{:}219 \times 10^9 \quad 2{:}069 \times 10^9 e$! (3)

(2)) $6878 = \frac{h^2}{(398600)} \frac{1}{1+e \cos 50^o}$

$h^2 = 2{:}742 \times 10^9 + 1{:}762 \times 10^9 e$! (4)

by (3) and (4), $3{:}219 \times 10^9 \quad 2{:}069 \times 10^9 e = 2{:}742 \times 10^9 + 1{:}762 \times 10^9 e$

$3{:}219 \quad 2{:}742 = e(1{:}762 \times 2{:}069)$

$e = 0{:}477{=}3{:}831$

$e = 0{:}1245$

Since e 1, it will be an ellipse.

$h^2 = 3{:}219 \times 10^9 \; 2{:}069 \times 10^9 (0{:}1245)$

$h^2 = 2961409500$

$h = 54418 \ km^2 / s$

c) $r_p = \frac{h_2}{} \frac{1}{1+e}$

$r_p = \frac{54418^2}{398600} \frac{1}{1+0{:}1245}$

r_p = 6607 km

d) $z_p = r_p \quad R_E$

z_p = 6607 6378

z_p = 229 km

e) $r_a = \frac{h_2}{1} \frac{1}{e}$

$r_a = \frac{54418^2}{398600} \frac{1}{1 \; 0{:}1245}$

r_a = 8486

$a = \frac{r_a + r_p}{2}$

$a = \frac{8486+6607}{2}$

a = 7546:5

3. At a given point of spacecraft's geocentric trajectory, the perigee alti-tude is 300 km, the speed is 15 km/s and the ight path angle is 50^o.Show that path is hyperbola and calculate

a) hyperbolic excess speed
b) angular momentum
c) true anomaly
d) eccentricity

Solution:

a) $r_p = R_E + z_p$ = 6378 + 300 = 6678

$v_{esc} = q\sqrt{\frac{2}{r}}$

$v_{esc} = q\sqrt{\frac{2(398600)}{6678}}$

v_{esc} = 10:925

$v_{inf^2} = v^2 - v_{esc}^2$

= 225 119:377

$v_{inf}^2 = 105:623$

$v_{inf} = 10:277\ km^2 = s^2$

b) $v_r = r \sin\ = 15 \sin 50^o = 11:49$

$v_p = r \cos\ = 15 \cos 50^o = 9:641$

$h = rv_p = 6678(9:461)$

$h = 64383\ km^2/s$

c) we know that $r = \frac{h_2}{} \frac{1}{1+e \cos}$

$6678 = \frac{64383^2}{398600} \frac{1}{1+e \cos}$

$e \cos = 0:5573\ !\ (1)$

$v_r = \frac{}{h} e \sin$

$11:49 = \frac{398600}{64393} e \sin$

$e \sin = 1:856\ !\ (2)$

$\tan = \frac{e \sin}{e \cos}$

$\tan = \frac{1:856}{0:5573}$

$= 73:28^o$

d) (1))$e \cos 73:28^o = 0:5573$

$e = 1:9379$

Since e 1, it will be hyperbolic orbit.

(5). The perigee of a satellite in a parabolic trajectory is 9000 km. Find the distance d between points P_1 and P_2 on orbit which are 11000km and 22000 km from the center of earth.

Solution: We know that, $r_p = \frac{h_2}{} \frac{1}{1+e}$

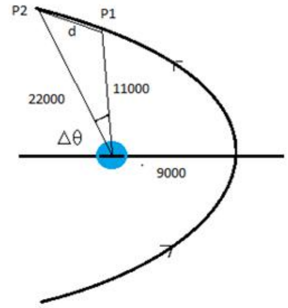

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Figure 1: parabola

Since it is a parabolic trajectory its eccentricity will be one.

$r_p = \frac{h^2}{\ } \frac{1}{1+1}$

$r_p = 2^{h2}$

$h = \sqrt{2 r_p}$

$h = \sqrt{2(398600)(9000)}$

$h = 84704 km^2/s$

True anomalies of points P_1 and P_2 are found using orbit equation,

we know that $r = \frac{h^2}{\ } \frac{1}{1+e\cos}$

$11000 = \frac{84704^2}{398600} \frac{1}{1+\cos{}_1}$

$_1 = 50{:}48^o$

$22000 = \frac{84704^2}{398600} \frac{1}{1+\cos{}_2}$

$_2 = 100{:}48^o$

$= {}_2 \qquad {}_1 = 100{:}48^o \quad 50{:}48^o$

$= 50^o$

From law of cosines from trignometry,

$d^2 = 11000^2 + 22000^2 \ 2(11000)(22000) \cos$

50^o $d = 17143$ km

4 REFERENCE

1. Orbital Mechanics 2nd Edition by John E. Prussing , Bruce A. Con-way.
2. Modern Astrodynamics Second Edition 2nd Edition by William E.
3. Fundamentals of Astrodynamics by Bate and Mueller.
4. Classical Mechanics by N C Rana and P S Joag.
5. Mechanics by L D Landau and E M Lifhitz.
6. Mathematical Methods of Classical Mechanics by V I Arnold.
7. Practical Conic Sections by J. W. Downs.
8. Coordinate Geometry By SK Goyal.