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A Technique for Partially Solving a Family of Diffusion Problems

Dr. Ayaz Ahmad
Assistant Professor, Institute of Technology, Patna, Bihar, India

ABSTRACT
Our aim in this paper is to expose the interesting role played by differ integral (specifically, semi derivatives and semi integrals)in solving certain diffusion problems. Along with the wave equation and Laplace equation, the diffusion equation is one of the three fundamental partial differential equation of mathematical physics. I will not discuss convential solutions of the diffusion equation at all. These range from closed form solutions for very simple model problems to computer methods for approximating the concentration of the diffusing substance on a network of points. Such solutions are described extensively in the literature .My purpose, rather, is to expose a technique for partially solving a family of diffusion problems, a technique that leads to a compact equation which is first order partially and half order temporally. I shall show that, for semi finite systems initially at equilibrium, our semi differential equation leads to a relationship between the intensive variable and the flux at the boundary. Use of this relationship then obviates the need to solve the original diffusion equation in those problems for which this behavior at the boundary is of primary importance. I shall, in fact, freely make use of the general properties established for differ integral operators as if all my functions were differ integrable.

Keywords: *Semi derivatives, semi integrals, Simple model problem, Semi infinite system and porous media.*

1. INTRODUCTION
Brief historical remarks: The science centre of the North American Rockwell Corporation where this work was initiated .But we particularly thanks Dr. E Richerd Cohen, Norman D Malmath, and Wyne M Robertson. Later phases of work ,carried out at Trent University ,were generously supported by the National Research council of Canada .We are indebted to John Belley ,John Berrett ,Pal COLE Irene Fitzpatrick, Karen Nolte, and Beverly White.

All scientists necessarily build on the foundations of those who have gone before. We are no exceptions, but our subject is vast and diverse and the distinguished list of our foterunners is correspondingly long .It would be individious to single out any group of them, nor can was claim to have always done justice to their ideas . Here we shall not discuss conventional of the diffusion equation at all. These range from closed form solution for simple model problems to computer methods for approximating the concentaction of the diffusing substance on a network of points. Such solutions are described extensively in the literature. Our purpose ,rather is to expose a technique for partially solving a family of diffusion equation ,a technique that leds to a compact equation which is first order spatially and half order temporally .We shall show that ,semi infinite systems initially at equilibirium ,our semi differential equation leads to a relationship between the intensive variable and the flux at the boundary . Use of this relationship then obviates the need to solve the original diffusion equation in those problems for which this behaviour at the boundary is of primary importance. As is inevitable when result proved generally are put to practice. Some of the restrictions that were found to be useful in establishing the theory may be difficult ,even impossible ,to verify in practice .No good scientist however would let this prevent him from applying the theory ,indeed application of the theory are frequently made without such verification ,and the results obtained

Preliminaries: The most powerful application of this theory ,namely to diffuse transport in a semi infinite medium Here a single equation __asserting the proportionality of a second order spatial partial derivative to a first order partial derivatives with respect to time ___governs a wide variety of transport phenomena ,heat in solid ,chemical species in homogeneous media ,vortices in fluids electricity in resistive capacitative lines , to maintion only a few .The replacement of this equation ,together with an initial and yhe asymptotic boundary condition ,by an equation linking a first order spatial derivative to a half order temporal derivative is the essence of this application , .The fundamental role played in this theory by semi differentiation and its inverse explains our preoccupation with the result of applying $\frac{d^{1/2}}{dx^{1/2}}$ and $\frac{d^{-1/2}}{dx^{-1/2}}$ to a wide variety of functions. We developed primarily to handle the semi differentiation and semi integration operators which arise in transport applications.

2. Transport in a semi infinite medium

Considering the diffusion equation $\frac{\partial}{\partial t}F(\rho,\sigma,\tau,t) = k\nabla^2 F(\rho,\sigma,\tau,t)$ …………………..2.1

In which Fis some intensive scalor quantity (temperature, concentration, vorticity, electrical potential, or the like)that varies with time and from point to point in a three dimensional homogeneous medium , while k is a constant appropriate to the medium and type the transport . In eqs (2.1)$\frac{\partial}{\partial t}$ effects the partial differentiation with respect to time and ∇^2 is the laplacian operator with respect to the spatial co ordinatesρ $,\sigma$ and $,\tau.$ When equation (2.1) is augmented by an initial condition and appropriate boundary geometric allows a reduction from three to one in number spatial coordinates needed to describe transport through the medium .We shall use three different values of a geometric factor g to characterize these simplifying geometries, namely

The convex sphere, g = 1

The convex cylinder, g = $\frac{1}{2}$: and

The plane, g = 0

The significance of the adjective "will be clarified by glancing at item 1 and noting that the boundary appears convex as viewed from the diffusion medium. The term "semi infinite" is commonly applied to the three cases. This is meant that the diffusion medium extends indefinitely in one direction from the boundary .In these geometries laplacian operator simplifies so that equation (2.1) becomes

$$\frac{\partial}{\partial t} F(r,t) - k\frac{\partial^2}{\partial t^2} F(r,t) - \frac{2gk}{r\pm R}\frac{\partial}{\partial r} F(r,t) = 0 \quad …….2.2$$

Encompassing the three values of g .Here r is the spatial coordinate directed normal to the boundary and having its origin at the boundary surfaces. In the case of spherical and cylindrical geometries, the R is without significance in the planar case.

The motive in restricting consideration to semi infinite geometries is that thereby the diffusion medium has only one boundary of concern, the other being "at finity" the same situation may be achieved with media that are less than infinite extent . (2.1)provided that the time domain is sufficiently restricted .As long as any perturbation which starts are at r= boundary at time zero does not approach any other boundary of the medium within times of interest.

The situation we shall treat are those in which the system is initially at equilibrium, so that $F(r,t) = F_0$,a constant ;

t < 0 , r≥ 0 ……………………………….2.3

At t =0 a perturbation of the system commences by some unspecified process occurring at the boundary .During times of interest , this perturbation does not affect region remote from the r =0 boundary ,so that the relationship Applies.

$$F(r,t) = F_0\ , \quad t \le r \le , \quad r = \begin{Bmatrix} \infty, & g = 0,\frac{1}{2},1 \\ R, & g = \frac{-1}{2},-1 \end{Bmatrix} \quad 2.4$$

Thus our problem is described by the partial differential equation (2.2) initial condition (2.3); and asymptotic condition (2.4) .It may be shown (Oldham and spanier, 1972; Oldham, 1973b) that the single equation

$$\frac{\partial}{\partial r} F(r,t) + \frac{1}{\sqrt{k}}\frac{\partial^{1/2}}{\partial t^{1/2}}[F(r,t) - F_0] + \frac{g}{r+R}[F(r,t) - F_0] = 0 \quad ……………………..2.5$$

Describes the problem equally well .Equation (2.5)is exact in the g=1 or cases ,and represents a short times approximation for g= $^1/_2$,$-^1/_2$ or -1. The derivation of the equation (2.5)is given in the next

section for the planar g= 0 case.
3Main result

$\frac{\partial}{\partial t} F(r,t) - k\frac{\partial^2}{\partial t^2} F(r,t) - \frac{2gk}{r\pm R}\frac{\partial}{\partial r} F(r,t) = 0 \ldots\ldots A$

Equation (A) describes the diffusion of an entity in the absence of sources and sinks; where there is no creation or annihiliation of the diffusing substances within the medium. Many practical problems however requiring the inclusion of volume sources or sinks. We now turn our attention to this subject

The equation

$\frac{\partial}{\partial t} F(r,t) - k\frac{\partial^2}{\partial t^2} F(r,t) = S - k\, F(r,t)$ …….3.1

Replaces g =0 instance of equation (A) when the transport is accompanied by a constant source S and a first order removal process ,embodied in the $k\,F(r,t)$ term . We shall assume that the uniform steady-state condition

$F(r,t) = \frac{S}{k}$t≤ 0 …………….3.2

Is in effect prior to the time t =0 .We shall sketch derivation of the relationship

$F(0,t) = \frac{S}{k} - \sqrt{k}e^{-kt}\frac{d^{\frac{-1}{2}}}{dt^{\frac{-1}{2}}}\left[e^{kt}\frac{d}{dr}F(0,t)\right]$ ……..3.3

Between the boundary value $F(0,t)$of the intensive variable and the boundary value of the flux, which is proportional to the term $-dF(0,t)/dr$.

Upon laplace transformation of equation (3.1), we obtain the equation

$$s\bar{F}(r,s) - F(r,o) - k\frac{\partial^2}{\partial r^2}\bar{F}(r,s) = \frac{S}{s} - k\bar{F}(r,s)$$

Use of our initial condition (3.2) leads to the ordinary differential equation

$\frac{d^2}{dr^2}\bar{F}(r,s) - \frac{s+k}{k} + \frac{S}{k}\left[\frac{s+k}{sk}\right] = 0$ ……….3.4

In the transform of F, the most general solution of equation (3.4) is

$\bar{F}(r,s) - \frac{S}{sk} =$

$p(s)exp\left(-r\sqrt{\frac{s+k}{k}}\right)p_1(s)exp\left(r\sqrt{\frac{s+k}{k}}\right)$ ….3.5

P(s) and $p_1(s)$ are arbitrary function of s ,if the geometory is semi infinite ,which we now assume the physical requirement that F remain bounded as r tends to infinity demands that $p_1(s) = 0$ .Eliminating the function $p_1(s)$ between equation obtained from it upon differentiation with respect to r ,we find

$$\bar{F}(r,s) = \frac{S}{sk} - \left(\sqrt{\frac{k}{k+s}}\right)\frac{d}{dr}\bar{F}(r,s)$$

It now requires only Laplace inversion and specialization to r=0 to obtain equation (3.3)

An interesting application of the preceding theory arising in modelling diffusion of atmospheric pollutant .We now describe such a model problem .

Consider a vertical column of unstirred air into base of which a pollutant commences to be injected at time t=0.Priror to t=0 the air unpolluted, so that

C(r, 0) = 0

Where C(r,t)denotes the pollutant concentration at height r at time t. After t =0 the rate of pollutant injection is some unspecified function J (t) of time. The pollutant reacts with air by some chemical reaction that is first order (or pseudo first order) with a rate constant k. The pollutant diffuses through air with a diffusion coefficient D that is assumed independent of height and of concentration .There are no volume source of pollutants, we seek to relate the ground –level pollutant concentration

C(t) = C(0,t) to the input flux J(t)

Identifying C(0,t) with F(0,t) and D with k in equation (3.3) and setting S=0 and J(0,t) = $-D\,dC(0,t)/dr$,we describe that

C(t) = $\frac{exp(-kt)}{\sqrt{D}}\frac{d^{\frac{-1}{2}}}{dt^{\frac{-1}{2}}}\{e^{(kt)}J(t)\}$

The generally of equation (3.6) is worth emphasing , It enables the ground concentration of pollutant to be predicted from the rate of pollutant generation for any time dependent J(t) .Similarly ,the inverse of equation (3.6)

J(t) = $\sqrt{D}$ exp(-kt)$\frac{d^{\frac{1}{2}}}{dt^{\frac{1}{2}}}\{e^{(kt)}C(t)\}$ …….3.7

Permits the generation rate to be constructed from a record of the time variation of ground pollution levels. By using equations (3.6).Consider the case where rate J(t) zero prior to t=0 ,is a constant J thereafter. Then

C(t) = $\frac{exp(kt)}{\sqrt{D}}\frac{d^{\frac{-1}{2}}}{dt^{\frac{-1}{2}}}\{Je^{(kt)}\} = \frac{J\ erf(\sqrt{kt})}{\sqrt{Dk}}$

Which shows that the pollutant concentration will rise to reacha final constant level of $\frac{J}{\sqrt{Dk}}$ and that the time to reach one half of the final level is 0 .23/k

As a more realistic example, consider the pollution generation rate to be sinusoidal with a mean value J and a minimum value of zero

$$J(t) = J + J\, sin(at)$$

With a equal to $\frac{\pi}{[12\ hours]}$and t= 0 corresponding to 9.00 A.M., this could represent a diurnal variation in pollution generation, typified by auto mobile traffic .Introduction of equation (3.8) into (3.6) followed by the indicated semi integration leads eventually to the result

$$C(t) = \frac{J\ erf(\sqrt{kt})}{\sqrt{Dk}} - \frac{Je^{-Kt}}{\sqrt{D(K^2+a^2)}}\{\beta\ Rew([\beta + i\alpha]\sqrt{t}) + \alpha\ Imw([\beta + i\alpha])\sqrt{t}\} + \frac{J}{\sqrt{D(K^2+a^2)}}[\beta\ cos(at) - \alpha sin(at)] \quad ....3.9$$

Where α and β are positive quantities defined by

$2\alpha^2 = \sqrt{(K^2 + a^2)}$ +k ,

$2\beta^2 = \sqrt{(K^2 + a^2)}$ - k ,

and Rew ( ) and Imw ( )are the real and imaginary parts of the complex error function of Faddeeva and Trent ev (1961)

As t becomes large ,the transient terms within equation (3.90vanish and leave

$$C(t \to \infty) = \frac{J}{\sqrt{D}}\left[\frac{1}{\sqrt{K}} + \frac{1}{(K^2+a^2)^{\frac{1}{4}}} sin\left(at - arc\ tan\left(\frac{\beta}{\alpha}\right)\right)\right]$$

This relationship gives

$$\frac{J}{\sqrt{D}}\left[\frac{1}{\sqrt{K}} + \frac{1}{(K^2+a^2)^{\frac{1}{4}}}\right]$$

As the extreme pollutant concentration, with the peak level occurring at some time between 3.00 and 6.00pm

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