International Journal of Trend in Scientific Research and Development (IJTSRD)
International Open Access Journal
ISSN No: 2456 - 6470 | www.ijtsrd.com | Volume - 2 | Issue – 1

Analyzation of Weak Convergence on L^p Space

R. Ranjitha
Research Scholar, Department of Mathematics, Vivekanandha College of Arts and Sciences for Women, Tiruchengode, Tamilnadu, India

S. Dickson
Assistant Professor, Department of Mathematics, Vivekanandha College for Women, Tiruchengode, Tamilnadu, India

J. Ravi
Assistant Professor, Department of Mathematics, Vivekanandha College for Women, Tiruchengode, Tamilnadu, India

ABSTRACT

In this paper a study on duality, weak convergences and weak sequential compactness on L^p space. A new generalized L^p space is investigated from the weak sequential convergence. Some more special types and properties of L^p are presented. Few relevant examples are also included to justify the proposed notions.

Keywords: *Dual space, Linear functional, Weak Convergence, Compact*

1. INTRODUCTION

The L^p spaces were introduced by FrigyesRiesz [9] as part of a program to formulate for functional and mappings defined on infinite dimensional spaces appropriate versions of properties possessed by functionals and mappings defined on finite dimensional spaces. In this paper, we study the weak convergence and weak compactness on the so-called Lebesgue spaces (or L^p – spaces as they are usually called).

2. PRELIMINARIES

Definition 2.1

Let X be a measure space. The measurable function $f$is said to be in L^p if it is p-integrable, that is, if $\int |f|^p \, d\mu < \infty$. The L^p norm of f is defined by $\|f\|_{L^p} = \left(\int |f|^p \, d\mu\right)^{1/p}$

Definition 2.2

Let X be a linear space and let $f \in X$. The ***norm*** of f is denoted by $\|f\|$ and it is defined as, $\|f\| = \sqrt{(f, f)}$.

Definition 2.3

Let X be a linear space. A real-valued functional$\|\,.\,\|$on X is called a norm provided for each f and g in X and each real number α,

i) $\|f\| \geq 0$ and $\|f\| = 0$ if and only if $f = 0$,

ii) $\|f+g\| \leq \|f\| + \|g\|$,

iii) $\|\alpha f\| = |\alpha| \|f\|$.By a ***normed linear space*** we mean a linear space together with a norm.

Definition 2.4

Let $f : X \to \mathbf{R}$ be a measurable function on a measure space(X, A, μ). The *essential* ***supremem*** of f on X is $\underset{X}{ess\sup} f = \inf\{a \in \mathbb{R} : \mu[x \epsilon X : f(x) > a] = 0\}$.

Equivalently, $\underset{X}{ess\sup} f = \inf\left\{\sup_X g : \quad g = f \text{ po int wise} a.e.\right\}$

Thus, the essential supremum of a function depends only on its μ-a.e. equivalence class.

Definition 2.4
Let (X, A, μ) be a measure space. The space $L^{\infty}(X)$ consists of point wise a.e.- equivalence classes of ***essentially bounded*** measurable functions $f : X \to \mathbf{R}$ with norm $\|f\|_{L^{\infty}} = ess\sup_{X}|f|$. In future, we will write ess sup f =sup f .

Definition 2.5
If p and q are positive real numbers such that $p+q = pq$ or equivalently, $\frac{1}{p}+\frac{1}{q}=1$, $(1 < p < \infty, 1 < q < \infty)$.Then we call p and q a pair of ***conjugate exponents.***

Definition 2.6
A sequence of function $\langle f_n \rangle$ is said to ***converge to*** f ***in the mean of order P*** if each f_n belongs to L^P and $\|f - f_n\|_p \to 0$.

Note:
Convergence in L^{∞} is nearly uniform convergence.

Definition 2.7
A ***linear functional*** on a linear space X is a real-valued function T on X such that for g and h in X and α and β real numbers,
$\mathrm{T}(\alpha.g + \beta.h) = \alpha.\mathrm{T}(g) + \beta.T(h)$.

Note: 2.8
1. The collection of linear functionals on a linear space is itself a linear space.
2. A linear functional is bounded if and only if it is continuous.

Definition 2.9
Let X and Y be a normed linear spaces. An ***isometric isomorphism*** of X into Y is a one-one linear transformation $f : X \to Y$ such that,
$\|f(x)\| = \|x\|$, for all $x \in X$.

3. WEAK SEQUENTIAL CONVERGENCE ON L^p SPACE

Some basic inequalities and results have been discussed.

Young's inequality3.1
Let $1 < p < \infty$ and let q be defined by $\frac{1}{p}+\frac{1}{q}=1$.Then, for any $a, b \geq 0$, we have $ab \leq \frac{a^p}{p}+\frac{b^q}{q}$, with equality occurring if and only if $a^{p-1} = b$.

Holder inequality 3.2
Let $1 < p < \infty$, and let q be defined by $\frac{1}{p}+\frac{1}{q}=1$.If $f \in L_p(E)$ and $g \in L_q(E)$
then $fg \in L_1(E)$ and $\left|\int_E fg\right| \leq \int_E |fg| \leq \|f\|_p \|g\|_q$.

Lemma 3.3
Let $1 < p < \infty$. If $f, g \in L_p$, then $f + g \in L_p$ and $\|f+g\|_p^p \leq 2^p\left(\|f\|_p^p + \|g\|_p^p\right)$. consequently, L^P is a vector space.

Minkowski's Inequality 3.4
Let $1 \leq p < \infty$ and let $f, g \in L_p$ .Then, $f + g \in L_p$ and $\|f+g\|_p \leq \|f\|_p + \|g\|_p$.

Definition 3.5
Let X be a normed linear space. Then the collection of bounded linear functionals on X is a linear space on which $\|.\|_*$ is a norm.This normed linear space is called the ***dual space*** of X and denoted by X^*.

Definition 3.6
Let X be a normed linear space. A sequence $\{f_n\}$ in X is said to ***converges weakly*** in X to f in X provided $\lim_{n\to\infty} T(f_n) = T(f)$ for all $T \in X^*$.We write $\{f_n\} \longrightarrow f$ in X to mean that f and each f_n belong to X and $\{f_n\}$ converges weakly in X to f .

Definition 3.7
A subset K of a normed linear space X is said to be ***weakly sequentially compact*** in X provided every sequence $\{f_n\}$ in K has a subsequence that converges weakly to $f \in K$.

Theorem 3.8

Let E be a measurable set and $1 \le p < \infty$. Suppose $\{f_n\} \to f$ in $L^p(E)$. Then $\{f_n\}$ is bounded in $L^p(E)$ and $\|f\|_p \le \lim \inf \|f_n\|_p$.

Proof:

Let q be a conjugate of p and f^* the conjugate function of f. We first establish the right-hand side on the above inequality. We infer from Holder's Inequality that

$$\int_E f^* . \; f_n \le \|f^*\|_q . \|f_n\|_p \text{ for all } n.$$

$$\int_E f^* . \; f_n \le \|f_n\|_p \text{ for all } n.$$

Since $\{f_n\}$ converges weakly to f and f^* belongs to $L^q(E)$,

$$\|f\|_p = \int_E f^* . \; f_n = \lim_{n\to\infty} \int_E f^* . \; f_n$$

$$\|f\|_p \le \lim \inf \|f_n\|_p$$

By contradiction to show that $\{f_n\}$ is bounded in $L^p(E)$. Assume $\{\|f_n\|_p\}$ is unbounded. Without loss of generality, by possibly taking scalar multiples of a subsequence, we suppose

$$\|f_n\|_p = n.3^n \text{ for all } n \Rightarrow \frac{1}{3^n}\|f_n\|_p = n \text{ forall } n . \longrightarrow ①$$

We inductively select a sequence of real numbers $\{\varepsilon_k\}$ for which $\varepsilon_k = \pm \frac{1}{3^k}$ for each k. Define $\varepsilon_1 = \frac{1}{3}$. If n is a natural number for which $\varepsilon_1, \varepsilon_2, \ldots . \varepsilon_n$ have been defined. Define

$$\varepsilon_{n+1} = \frac{1}{3^{n+1}} \text{ if} \int_E \left[\sum_{k=1}^{n} \varepsilon_k (f_n)^* \right] . f_{n+1} \ge 0 \text{ and } \varepsilon_{n+1} = -\frac{1}{3^{n+1}} \text{ if}$$

the above integral is negative.

Therefore, by ① and the definition of conjugate function,

$$\left| \int_E \left[\sum_{k=1}^{n} \varepsilon_k (f_k)^* \right] . f_n \right| \ge \frac{1}{3^n} \|f_n\|_p$$

$$\left| \int_E \left[\sum_{k=1}^{n} \varepsilon_k . (f_k)^* \right] . f_n \right| \ge n \longrightarrow ②$$

and $\left\|\varepsilon_n . (f_n)^*\right\|_q = \frac{1}{3^n}$ for all n.

Since $\left\|\varepsilon_k . (f_k)^*\right\|_q = \frac{1}{3^k}$ for all k, the sequence of partial sums of the series $\sum_{k=1}^{\infty} \varepsilon_k . (f_k)^*$ is a Cauchy sequencein $L^q(E)$.

We know that, "If X is a measure space and $1 \le p \le \infty$, then $L^p(X)$ is complete" [2] tells us that $L^q(E)$ is complete.

Define the function $g \in L^q(E)$ by $g = \sum_{k=1}^{\infty} \varepsilon_k . (f_k)^*$.

Fix a natural number n.

We infer from the triangle inequality, ②, and Holder's Inequality that

$$\left| \int_E g . f_n \right| = \left| \int_E \left[\sum_{k=1}^{\infty} \varepsilon_k . (f_k)^* \right] . f_n \right|$$

$$= \left| \left(\int_E \left[\sum_{k=1}^{n} \varepsilon_k . (f_k)^* \right] . f_n \right) - \left(\int_E \left[\sum_{k=n+1}^{\infty} \varepsilon_k . (f_k)^* \right] . f_n \right) \right|$$

$$\ge \left| \int_E \left[\sum_{k=1}^{n} \varepsilon_k . (f_k)^* \right] . f_n \right| - \left| \int_E \left[\sum_{k=n+1}^{\infty} \varepsilon_k . (f_k)^* \right] . f_n \right|$$

$$\ge n - \left| \int_E \left[\sum_{k=n+1}^{\infty} \varepsilon_k . (f_k)^* \right] . f_n \right|$$

$$\ge n - \left[\sum_{k=n+1}^{\infty} \frac{1}{3^k} \right] . \|f_n\|_p = n - \frac{1}{3^n} . \frac{1}{2} . \|f_n\|_p$$

$$\left| \int_E g . f_n \right| = \frac{n}{2}.$$

This is a contradiction because, the sequence $\{f_n\}$ converges weakly in $L^p(E)$ and g belongs to $L^q(E)$.

The sequence of real numbers $\left\{ \int_E g . f_n \right\}$ converges and therefore is bounded.

Hence $\{f_n\}$ is bounded in L^p.

Corollary 3.9

Let E be a measurable set, $1 \le p < \infty$ and q the conjugate of p. Suppose $\{f_n\}$ converges weakly to f in $L^p(E)$

and $\{g_n\}$ converges strongly to g in $L^q(E)$. Then $\lim_{n\to\infty}\int_E g_n.f_n = \int_E g.f$

Proof:

For each index n

$$\int_E g_n.f_n - \int_E g.f = \int_E g_n.f_n - \int_E g.f_n + \int_E g.f_n - \int_E g.f$$

$$= \int_E [g_n - g].f_n + \int_E g.f_n - \int_E g.f$$

According to the preceding theorem, there is a constant $C \geq 0$ for which $\|f_n\|_p \leq C$ for all n.

Consider $\left|\int_E g_n.f_n - \int_E g.f\right| = \left|\int_E [g_n - g].f_n + \int_E g.f_n - \int_E g.f\right|$

Therefore, by Holder's Inequality,

$$\left|\int_E g_n.f_n - \int_E g.f\right| \leq \left|\int_E [g_n - g].f_n\right| + \left|\int_E g.f_n - \int_E g.f\right|$$

$$\left|\int_E g_n.f_n - \int_E g.f\right| \leq C.\|g_n - g\|_q + \left|\int_E g.f_n - \int_E g.f\right| \text{ for all } n$$

Taking limit on both sides, we get

$$\lim_{n\to\infty}\left|\int_E g_n.f_n - \int_E g.f\right| \leq \lim_{n\to\infty}\left[C.\|g_n - g\|_q + \left|\int_E g.f_n - \int_E g.f\right|\right]$$

$$\lim_{n\to\infty}\left|\int_E g_n.f_n - \int_E g.f\right| \leq C.\lim_{n\to\infty}\|g_n - g\|_q + \left|\lim_{n\to\infty}\int_E g.f_n - \int_E g.f\right| \longrightarrow ①$$

From these inequalities and the fact that both $\lim_{n\to\infty}\|g_n - g\|_q = 0$ and $\lim_{n\to\infty}\int_E g.f_n = \int_E g.f$.

Substitute above inequalities in equation ①, we get

$$\left|\lim_{n\to\infty}\int_E g_n.f_n - \int_E g.f\right| \leq C.0 + \left|\int_E g.f - \int_E g.f\right|$$

$$\left|\lim_{n\to\infty}\int_E g_n.f_n - \int_E g.f\right| = 0$$

$$\lim_{n\to\infty}\int_E g_n.f_n - \int_E g.f = 0$$

It follows that $\lim_{n\to\infty}\int_E g_n.f_n = \int_E g.f$.

Proposition 3.10

Let E be a measurable set, $1 \leq p < \infty$, and q the conjugate of p. Assume $\mathcal{F}$ is a subset of $L^q(E)$ whose linear span is dense in $L^q(E)$. Let $\{f_n\}$ be a bounded sequence in $L^p(E)$ and f belong to $L^p(E)$. Then $\{f_n\} \to f$ in $L^p(E)$ if and only if $\lim_{n\to\infty}\int_E f_n.g = \int_E f.g$ for all $g \in \mathcal{F}$. $\longrightarrow ①$

Proof:

Assume that ① holds. To verify weak convergence, let g_0 belong to $L^q(E)$.

We show that $\lim_{n\to\infty}\int_E f_n.g_0 = \int_E f.g_0$. Let $\varepsilon > 0$. We must find a natural number N for which

$$\left|\int_E f_n.g_0 - \int_E f.g_0\right| < \varepsilon \text{ if } n \geq N. \longrightarrow ②$$

Observe that for any $g \in L^q(E)$ and natural number n,

$$\int_E f_n.g_0 - \int_E f.g_0 = \int_E f_n.g_0 - \int_E f.g_0 - \int_E f_n.g + \int_E f_n.g + \int_E f.g - \int_E f.g$$

$$= \int_E f_n.g_0 - \int_E f.g_0 - \int_E f_n.g + \int_E f.g + \int_E f_n.g - \int_E f.g$$

$$= \int_E (f_n - f).g_0 - \int_E (f_n - f).g + \int_E (f_n - f).g$$

$$\int_E f_n.g_0 - \int_E f.g_0 = \int_E (f_n - f).(g_0 - g) + \int_E (f_n - f).g$$

$$\left|\int_E f_n.g_0 - \int_E f.g_0\right| = \left|\int_E (f_n - f).(g_0 - g) + \int_E (f_n - f).g\right|$$

and therefore, by using Holder's inequality,

$$\left|\int_E f_n.g_0 - \int_E f.g_0\right| \leq \|f_n - f\|_p.\|g_0 - g\|_q + \left|\int_E f_n.g - \int_E f.g\right|.$$

Since $\{f_n\}$ is bounded in $L^p(E)$ and the linear span of is dense in $L^q(E)$, there is a function g in this linear span for which

$$\|f_n - f\|_p.\|g_0 - g\|_q < \frac{\varepsilon}{2} \text{ for all } n.$$

We infer from ①, the linearity of integration, and the linearity of convergence for sequences of real numbers, that

$$\lim_{n\to\infty}\int_E f_n.g = \int_E f.g.$$

Therefore, there is a natural number N for which

$\left|\int_E f_n.g - \int_E f.g\right| < \frac{\varepsilon}{2}$ if $n \geq N$. It is clear that ② holds for this choice of N. Therefore $\lim_{n\to\infty}\int_E f_n.g = \int_E f.g$ is true for all $g \in \mathcal{F}$.

4. DUALITY ON L^p SPACE

Theorem 4.1

Suppose that (X, A, μ) is a measure space and $1 < p \le \infty$. If $f \in L^{p^1}(X)$, then $F(g) = \int fg \quad d\mu$, defines a bounded linear functional $F : L^p(X) \to \mathbf{R}$, and $\|F\|_{L^{p*}} = \|f\|_{L^{p^1}}$. If X is a $\sigma-$finite, then the same result holds for $p = 1$.

Proof:

From Holder's inequality, we have for $1 \le p \le \infty$ that $|F(g)| \le \|f\|_{L^{p^1}} \|g\|_{L^p}$ which implies that F is a bounded linear functional on L^P with $\|F\|_{L^{p*}} \le \|f\|_{L^{p^1}}$.

In proving the reverse inequality, we may assume that $f \ne 0$ (otherwise the result is trivial).

First, suppose that $1 < p < \infty$.

Let $g = (\text{sgn } f)\left(\frac{|f|}{\|f\|_{L^{p^1}}}\right)^{p^1/p}$. Then $g \in L^p$, since $f \in L^{p^1}$, and $\|g\|_{L^p} = 1$.

Also, since $\frac{p^1}{p} = p^1 - 1$,

$$F(g) = \int (\text{sgn } f) f \left(\frac{|f|}{\|f\|_{L^{p^1}}}\right)^{p^1-1} d\mu = \|f\|_{L^{p^1}}.$$

Since $\|g\|_{L^p} = 1$, we have $\|F\|_{L^{p*}} \ge |F(g)|$, so that $\|F\|_{L^{p*}} \ge \|f\|_{L^{p^1}}$.

If $p = \infty$, we get the same conclusion by taking $g = \text{sgn } f \in L^\infty$. Thus, in these cases the supremum defining $\|F\|_{L^{p*}}$ is actually attained for a suitable function g.

Suppose that $p = 1$ and X is $\sigma-$finite. For $\varepsilon > 0$, let $A = \{x \in X : |f(x)| > \|f\|_{L^\infty} - \varepsilon\}$.

Then $0 < \mu(A) \le \infty$. Moreover, since X is $\sigma-$finite, there is an increasing sequence of sets A_n of finite measure whose union is A such that $\mu(A_n) \to \mu(A)$. So we can find a subset $B \subset A$ such that $0 < \mu(B) < \infty$.

Let $g = (\text{sgn } f)\frac{\chi_B}{\mu(B)}$. Then $g \in L^1(X)$ with $\|g\|_{L^1} = 1$, and

$$F(g) = \frac{1}{\mu(B)} \int_B |f| \, d\mu \ge \|f\|_{L^\infty} - \varepsilon.$$

It follows that $\|F\|_{L^{1*}} \ge \|f\|_{L^\infty} - \varepsilon$, therefore $\|F\|_{L^{1*}} \ge \|f\|_{L^\infty}$ since $\varepsilon > 0$ is arbitrary.

This theorem shows that the map $F = J(f)$ defined by $J : L^{p^1}(X) \to L^p(X)^*$,

$J(f) : g \to \int fg \quad d\mu$, Is a isometry from L^{p^1} into L^{p*}.

Note: 4.2

The following result is that J is onto when $1 < p < \infty$, meaning that every bounded linear functional on L^P arises in this way from an L^{p^1} - function. The proof is based on that if $F : L^p(X) \to \mathbf{R}$ is a bounded linear functional on $L^p(X)$, then $\nu(E) = F(\chi_E)$ defines an absolutely continuous measure on (X, A, μ).

Lemma4.3

Let (X, A, μ) be a $\sigma-$finite measure space and $1 \le p < \infty$. For f an integrable function over X, suppose there is an $M \ge 0$ such that for every simple function g on X that vanishes outside of a set of finite measure, $\left|\int_X fg \quad d\mu\right| \le M.\|g\|_p$ ....(1). Then f belongs to $L^q(X, \mu)$, where q is conjugate of p. Moreover, $\|f\|_q \le M$.

Proof:

First consider the case $p > 1$. Since $|f|$ is a non-negative measurable function and the measure space is $\sigma-$finite, according to the Simple Approximation Theorem, there is a sequence of simple functions $\{\varphi_n\}$, each of which vanishes outside of a set of finite measure, that converges point wise on X to $|f|$ and $0 \le \varphi_n \le |f|$ on E for all n.

Since $\{\varphi_n^q\}$ converges point wise on X to $|f|^q$.

We know that," If (X, A, μ) is a measure space and $\{f_n\}$ a sequence of non-negative measurable functions on X for which $\{f_n\} \to f$ point wise a.e. on X and if f is measurable. Then $\int_X f \quad d\mu \le \liminf \int_X f_n d\mu$." [3]

The above statement tells us that to show that $|f|^q$ is integrable and $\|f\|_q \le M$ it suffices to show that

$\int_X \varphi_n^q d\mu \le M^q$ for all n……...............................(2)

Fix a natural number n. To verify (1), we estimate the functional values of φ_n^q as follows:

$\varphi_n^q = \varphi_n . \varphi_n^{q-1} \le |f| . \varphi_n^{q-1}$

$\varphi_n^q = f . \operatorname{sgn}(f) . \varphi_n^{q-1}$ on X…............................(3)

Define the simple function g_n by $g_n = \operatorname{sgn}(f) . \varphi_n^{q-1}$ on X.

We infer from (1) and (2) that

$\int_X \varphi_n^q d\mu \le \int_X f . g_n d\mu$

$\int_X \varphi_n^q d\mu \le M \|g_n\|_p$. ….....................................(4)

Since p and q are conjugate, $p(q-1) = q$ and therefore $\int_X |g_n|^p \, d\mu = \int_X \varphi_n^{p(q-1)} d\mu = \int_X \varphi_n^q d\mu$

Thus we may rewrite (4) as $\int_X \varphi_n^q d\mu \le M . \left[\int_X \varphi_n^q d\mu \right]^{1/p}$

For each n, φ_n^q is a simple function that vanishes outside of a set of finite measure and therefore it is integrable. Thus the preceding integral inequality may be rewritten as

$\left[\int_X \varphi_n^q d\mu \right]^{1-1/p} \le M$.

Since $1 - \frac{1}{p} = \frac{1}{q}$, we have verified (2).

It remains to consider the case $p = 1$. We must show that M is an essential upper bound for f .

We argue by contradiction.

If M is not an essential upper bound, then there is some ∈>0 for which the set $X_\varepsilon = \{x \in X : |f(x)| > M + \varepsilon\}$ has nonzero measure.

Since X is $\sigma-$finite, we may choose a subset of X_ε with finite positive measure.

If we let g be the characteristic function of such a set we contradict (1).

5. WEAK SEQUENTIAL COMPACTNESS

Theorem 5.1

Let X be a separable normed linear space and $\{T_n\}$ a sequence in its dual space X^* that is bounded, that is, there is an $M \ge 0$ for which $|T_n(f)| \le M . \|f\|$ ……(1) for all f in X and all n.Then there is subsequence $\{T_{n_k}\}$ of $\{T_n\}$ and T in X^* for which $\lim_{k\to\infty} T_{n_k}(f) = T(f)$ ….(2) for all f in X .

Proof:

Let $\{f_j\}_{j=1}^{\infty}$ be a countable subset of X that is dense in X .We infer from (1) that the sequence of real numbers $\{T_n(f_1)\}$ is bounded. Therefore, by the Bolzano-WeierstrassTheorem,"Every bounded sequence of real numbers has a convergent subsequence." there is a strictly increasing sequence of integers $\{s(1,n)\}$ and a number a_1 for which $\lim_{n\to\infty} T_{s(1,n)}(f_1) = a_1$.

We again use (1) to conclude that the sequence of real numbers $\{T_{s(1,n)}(f_2)\}$ is bounded and so again by the Bolzano-Weierstrass Theorem, there is a subsequence $\{s(2,n)\}$ of $\{s(1,n)\}$ and a number for a_2 which $\lim_{n\to\infty} T_{s(2,n)}(f_2) = a_2$ .We inductively continue this selection process to obtain a countable collection of strictly increasing sequences of natural numbers $\{\{s(\mathrm{j},n)\}\}_{j=1}^{\infty}$ and a sequence of real numbers $\{a_j\}$ such that for each j, $\{s(\mathrm{j}+1,n)\}$ is a subsequence of $\{s(\mathrm{j},n)\}$, and $\lim_{n\to\infty} T_{s(\mathrm{j},n)}(f_j) = a_j$.

For each index k, define $n_k = s(k,k)$. Then for each j, $\{n_k\}_{k=j}^{\infty}$ is a subsequence of $\{s(\mathrm{j},\mathrm{k})\}$ and hence $\lim_{k\to\infty} T_{n_k}(f_j) = a_j$ for all j .Since $\{T_{n_k}\}$ is bounded in X^* and $\{T_{n_k}(f)\}$ is a Cauchy sequence for each f is a dense subset of X , $\{T_{n_k}(f)\}$ is Cauchy for all f in X . The real numbers are complete. Therefore we may define

$T(f) = \lim_{k\to\infty} T_{n_k}(f)$ for all $f \in X$.

Since each T_{n_k} is linear, the limit functional T is linear.

Since $\left|T_{n_k}(f)\right| \le M.\|f\|$ for all k and all $f \in X$,

$\left|T(f)\right| = \lim_{k\to\infty}\left|T_{n_k}(f)\right| \le M.\|f\|$ for all $f \in X$

Therefore T is bounded.

Note 5.2:
The following theorem tells the existence of week sequential compactness on L^p space

Theorem 5.3
Let E be a measurable set and $1 < p < \infty$. Then every bounded sequence in $L^p(E)$ has a subsequence that converges weakly in $L^p(E)$ to a function in $L^p(E)$.

Proof:
Let q be the conjugate of p. Let $\{f_n\}$ be a bounded sequence in $L^p(E)$. Define $X = L^q(E)$.

Let n be a natural number. Define the functional T_n on X by $T_n(g) = \int_E f_n.g$ for g in $X = L^q(E)$.

We use the statement, " Let E be a measurable set, $1 \le p < \infty$, q the conjugate of p and g belong to $L^q(E)$. Define the functional T on $L^p(E)$ by

$T(f) = \int_E g.f$ for all $f \in L^p(E)$.

Then T is a bounded linear functional on $L^p(E)$ and $\|T\|_* = \|g\|_q$."

Now p and q interchanged and the observation that p is the conjugate of q, tells us that each T_n is a bounded linear functional on X and $\|T_n\|_* = \|f_n\|_p$

Since $\{f_n\}$ is a bounded sequence in $L^p(E)$, $\{T_n\}$ is a bounded sequence in X^*.

Moreover, according to " A function f on a closed, bounded interval $[a,b]$ is absolutely continuous on $[a,b]$ if and only if it is an indefinite integral over $[a,b]$." [3]

Since $1 < q < \infty$, $X = L^q(E)$ is separable.

Therefore, by "Let X be a separable normed linear space and $\{T_n\}$ a sequence in its dual space X^* that is bounded, that is, there is an $M \ge 0$ for which $\left|T_n(f)\right| \le M.\|f\|$ for all f in X and all n".

Then there is subsequence $\{T_{n_k}\}$ of $\{T_n\}$ and T in X^* for which $\lim_{k\to\infty} T_{n_k}(f) = T(f)$ for all f in X."There is a subsequence $\{T_{n_k}\}$ and $T \in X^*$ such that $\lim_{k\to\infty} T_{n_k}(g) = T(g)$ for all g in $X = L^q(E)$…(1)

The Riesz Representation Theorem, with p and q interchanged, tells us that there is a function f in $L^p(E)$ for which $T(g) = \int_E f.g$ for all g in $X = L^q(E)$.

But (1)means that $\lim_{k\to\infty}\int_E f_{n_k}.g = \int_E f.g$ for all g in $X = L^q(E)$. $\lim_{n\to\infty}\int_E g.f_n = \int_E g.f$

According to" Let E be a measurable set, $1 \le p < \infty$ and q the conjugate of p. Then $\{f_n\} \to f$ in $L^p(E)$ if and only iffor all $g \in L^q(E)$."

Therefore $\{f_{n_k}\}$ converges weakly to f in $L^p(E)$.

Hence every bounded sequence in $L^p(E)$ has a subsequence that converges weakly in $L^p(E)$ to a function in $L^p(E)$.

6. CONCLUSION
In this paper the duality, weak convergences and weak sequential compactness on L^p space has been discussed. A new generalized L^p space viz. theorem of compactness on L^p spaceare investigated. We hope that the space will be a powerful tool to study the various properties of lebesgue space and this endeavor is a small step towards that goal.

REFERENCES

1. **Novinger, W.P** [1992], Mean convergence in Lp spaces, Proceedings of the American Mathematical Society, 34, 627-628.
2. **Walter Rudin** [1966], Real and complex analysis, 3rd edition, Tata McGraw Hill, New York, 1987.
3. **Roydon,H.L**[1963], Real Analysis, 3rd edition, Macmillan, New York, 1988.

4. **Jain P.K, Gupta V.P** and **Pankaj Jain,** Lebesgue Measure And Integration, 2nd edition, New Age International (P) Limited, New Delhi.
5. **Carothers N.L,** Real Analysis, Cambridge University press, UK.
6. **Karunakaran .V,** Real Analysis, Pearson, India, www.pearsoned.co.in/vkarunakaran.
7. **Royden H.L,** and **Fitzpatrick P.M,** Real Analysis, 4th edition, Pearson India Education Services Pvt. Ltd. India, www.peaeson.co.in.
8. **FrigyesRiesz** and **BelaSz-Nagy,** Functional Analysis,Dover, 1990.