International Journal of Trend in Scientific Research and Development (IJTSRD)
International Open Access Journal
ISSN No: 2456 - 6470 | www.ijtsrd.com | Volume - 2 | Issue – 1

Limit Cycles Investigation for a Class of Nonlinear Systems via Differential and Integral Inequalities

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ABSTRACT

In this paper, the existence of limit cycles for a class of nonlinear systems is explored. Based on the time-domain approach with differential and integral inequalities, the phenomenon of the stable limit cycle can be accurately verified for such nonlinear systems. Furthermore, the exponentially stable limit cycles, frequency of oscillation, and guaranteed convergence rate can be correctly calculated. Finally, some numerical simulations are provided to demonstrate the feasibility and effectiveness of the main results.

Keywords: *Limit cycle, nonlinear systems, stable limit cycles, exponential convergence rate*

1. INTRODUCTION

Nonlinear network may cause oscillations with fixed period and fixed amplitude. These oscillations are named limit cycles, e.g., RLC electrical circuit with a nonlinear resistor and Van der Pol equation. Limit cycles are special phenomenon of nonlinear networks and have been widely investigated; see, for example, [1-12] and the references therein.

Prediction of limit cycles is very meaningful, in view of the fact that limit cycles can occur in any kind of physical system. Frequently, a limit cycle can be worthwhile. This is the case of limit cycles in the electronic oscillators utilized in factories and laboratories. There are at least four methods to explore the phenomenon of limit cycles, namely describing function technique, Poincare-Bendixson theorem, Piecewise-linearized methodology, and Lyapunov-like approach. The disadvantages of the describing function method are related to its approximate nature, and include the possibility of inaccurate predictions. Besides, the Poincare-Bendixson theorem only provides a necessary condition to ensure the existence of limit cycles. Therefore, even the conditions of the Poincare-Bendixson theorem are meted for a system, the existence of limit cycles cannot be guaranteed for such a system.

In this paper, based on the time-domain approach with differential and integral inequalities, the phenomenon of the stable limit cycle will be accurately verified for a class of nonlinear systems. Meanwhile, the exponentially stable limit cycles, frequency of oscillation, and guaranteed convergence rate will be calculated. At last, several numerical simulations will offered to show the feasibility and effectiveness of the obtained results.

2. PROBLEM FORMULATION AND MAIN RESULTS

In this paper, we consider the following nonlinear system:

$$\dot{x}_1(t) = -bx_3(t) - x_1(t)\left[x_2^2(t)+1\right] \cdot \left[x_1^2(t)+x_3^2(t)-a\right], \tag{1a}$$

$$\dot{x}_2(t) = x_1(t) + x_3(t), \tag{1b}$$

$$\dot{x}_3(t) = bx_1(t) - x_3(t)\left[x_2^2(t)+1\right] \cdot \left[x_1^2(t)+x_3^2(t)-a\right], \tag{1c}$$

$$x(0)^T := [x_{10} \quad x_{20} \quad x_{30}]^T, \qquad (1d)$$

Where $x(t) := [x_1(t) \quad x_2(t) \quad x_3(t)]^T \in \Re^{3\times 1}$ is the state vector, $[x_{10} \quad x_{20} \quad x_{30}]^T$ is the initial value, and $a, b \in \Re$ represent the parameters of the system, with $a > 0$. Clearly, $x = 0$ is an equivalent point of system (1), i.e., the solution of system (1) is given by $x(t) = 0$ if $x(0) = 0$. To avoid the apparent case of $x(0) = 0$, in the following, we only investigate the system (1) in case of $x(0) \neq 0$.

Definition 1

Consider the system (1). The closed and bounded manifold $s(x) = 0$, in the $x_1 - x_3$ plane, is said to be an exponentially stable limit cycle if there exist two positive numbers α and β such that the manifold of $s(x) = 0$ along the trajectories of system (1) meets the following inequality

$$|s(x(t))| \le \beta \cdot \exp[-\alpha(t - t_0)], \quad \forall t \ge t_0.$$

In this case, the positive number α is called the guaranteed convergence rate.

Now, we are in a position to present the main results for the existence of limit cycles of system (1).

Theorem 1.

All of phase trajectories of the system (1) tend to the exponentially stable limit cycle $s(x) = x_1^2 + x_3^2 - a = 0$ in the $x_1 - x_3$ plane, with the guaranteed convergence rate

$$\alpha := \begin{cases} \infty, & \text{if } x_{10}^2 + x_{30}^2 = a, \\ 2a, & \text{if } x_{10}^2 + x_{30}^2 > a, \\ 2(x_{10}^2 + x_{30}^2) & \text{if } x_{10}^2 + x_{30}^2 < a. \end{cases}$$

Besides, the states $x_1(t)$ and $x_3(t)$ exponentially track, respectively, the trajectories $\sqrt{a}\cos\left[bt + \tan^{-1}\left(\frac{x_{30}}{x_{10}}\right)\right]$ and $\sqrt{a}\sin\left[bt + \tan^{-1}\left(\frac{x_{30}}{x_{10}}\right)\right]$, in the time domain, with the guaranteed convergence rate $\alpha/2$.

Proof. Define a smooth manifold $s(x) = 0$ and a continuous function $\theta(x) := \tan^{-1}\left[\frac{x_3}{x_1}\right]$ with $s(x) = x_1^2 + x_3^2 - a$. Then the time derivatives of $s^2(x)$ and $\theta(x)$ along the trajectories of system (1) is given by

$$\frac{ds^2(x(t))}{dt} = 2s(x(t)) \cdot (2x_1\dot{x}_1 + 2x_3\dot{x}_3) = -4(x_1^2 + x_3^2)(x_2^2 + 1)s^2(x(t)). \qquad (2)$$

$$\frac{d\theta(x(t))}{dt} = \frac{\dot{x}_3 x_1 - \dot{x}_1 x_3}{x_1^2 + x_3^2} = b.$$

It follows that

$$\theta(x(t)) = bt + \tan^{-1}\left(\frac{x_{30}}{x_{10}}\right). \qquad (3)$$

In the following, there are three cases to discuss the trajectories of the system of (1).

Case 1: $x_1^2(0) + x_3^2(0) = a$ (or equivalently; $s(x(0)) = 0$)

In this case, from (2), it can be obtained that $\frac{ds^2(x(t))}{dt} = 0,$ which implies

$$x_1^2(t) + x_3^2(t) = a, \quad \forall\, t \ge 0. \qquad (4)$$

Hence we conclude that

$$x_1(t) = \sqrt{a}\cos\left[bt + \tan^{-1}\left(\frac{x_{30}}{x_{10}}\right)\right], \quad \forall\, t \ge 0,$$

$$x_2(t) = \sqrt{a}\sin\left[bt + \tan^{-1}\left(\frac{x_{30}}{x_{10}}\right)\right], \quad \forall\, t \ge 0,$$

$$s(x(t)) = 0, \quad \forall\, t \ge 0,$$

in view of (3) and (4).

Case 2: $x_1^2(0) + x_2^2(0) > a$ (or equivalently; $s(x(0)) > 0$)

In this case, from (2), it can be obtained that $s^2(x(t))$ is a strictly decreasing function of t with $s^2(x(t)) \ge 0, \forall\, t \ge 0$, and

$$\begin{aligned} \frac{ds^2(x(t))}{dt} &= -4[x_1^2(t) + x_3^2(t)][x_2^2(t) + 1]s^2(x(t)) \\ &\le -4[x_1^2(t) + x_3^2(t)]s^2(x(t)) \\ &= -4a\, s^2(x(t)), \quad \forall\, t \ge 0. \end{aligned}$$

Applying the Bellman-Gronwall inequality with above differential inequality, one has

$$s^2(x(t)) \le s^2(x(0)) \cdot \exp[-4at], \quad \forall\, t \ge 0,$$

This implies

$$|s(x(t))| \le |s(x(0))| \cdot \exp[-2at], \quad \forall\, t \ge 0,$$

$$\left|\sqrt{x_1^2(t)+x_3^2(t)}-\sqrt{a}\right|^2$$
$$\le \left|\sqrt{x_1^2(t)+x_3^2(t)}-\sqrt{a}\right|\cdot\left|\sqrt{x_1^2(t)+x_3^2(t)}+a\right|$$
$$=\left|x_1^2(t)+x_3^2(t)-a\right|$$
$$=\left|s(x(t))\right|$$
$$\le \left|s(x(0))\right|\cdot\exp[-2at],\quad \forall\, t\ge 0.$$

It yields

$$\left|\sqrt{x_1^2(t)+x_3^2(t)}-\sqrt{a}\right|$$
$$\le \sqrt{\left|s(x(0))\right|}\cdot\exp[-at],\quad \forall\, t\ge 0. \tag{5}$$

Consequently, by (3) and (5), we conclude that

$$\left|x_1(t)-\sqrt{a}\cos\left[bt+\tan^{-1}\left(\frac{x_{30}}{x_{10}}\right)\right]\right|$$
$$=\left|\sqrt{x_1^2(t)+x_3^2(t)}\cos\left[bt+\tan^{-1}\left(\frac{x_{30}}{x_{10}}\right)\right]\right.$$
$$\left.-\sqrt{a}\cos\left[bt+\tan^{-1}\left(\frac{x_{30}}{x_{10}}\right)\right]\right|$$
$$=\left|\sqrt{x_1^2(t)+x_3^2(t)}-\sqrt{a}\right|\cdot\left|\cos\left[bt+\tan^{-1}\left(\frac{x_{30}}{x_{10}}\right)\right]\right|$$
$$\le\left|\sqrt{x_1^2(t)+x_3^2(t)}-\sqrt{a}\right|$$
$$\le \sqrt{\left|s(x(0))\right|}\cdot\exp[-at],\quad \forall\, t\ge 0,$$

$$\left|x_3(t)-\sqrt{a}\sin\left[bt+\tan^{-1}\left(\frac{x_{20}}{x_{10}}\right)\right]\right|$$
$$=\left|\sqrt{x_1^2(t)+x_3^2(t)}\sin\left[bt+\tan^{-1}\left(\frac{x_{20}}{x_{10}}\right)\right]\right.$$
$$\left.-\sqrt{a}\sin\left[bt+\tan^{-1}\left(\frac{x_{20}}{x_{10}}\right)\right]\right|$$
$$=\left|\sqrt{x_1^2(t)+x_3^2(t)}-\sqrt{a}\right|\cdot\left|\sin\left[bt+\tan^{-1}\left(\frac{x_{20}}{x_{10}}\right)\right]\right|$$
$$\le\left|\sqrt{x_1^2(t)+x_3^2(t)}-\sqrt{a}\right|$$
$$\le \sqrt{\left|s(x(0))\right|}\cdot\exp[-at],\quad \forall\, t\ge 0.$$

Case 3: $x_1^2(0)+x_3^2(0)<a$ (or equivalently; $s(x(0))<0$)

In this case, from (2), it can be obtained that $s^2(x(t))$ is a strictly decreasing function of t with $s^2(x(t))\ge 0,\ \forall\, t\ge 0$, and

$$\frac{ds^2(x(t))}{dt}=-4\left[x_1^2(t)+x_3^2(t)\right]\left[x_2^2(t)+1\right]s^2(x(t))$$
$$\le -4\left[x_1^2(t)+x_3^2(t)\right]s^2(x(t))$$
$$\le -4\left(x_{10}^2+x_{30}^2\right)s^2(x(t)),\quad \forall\, t\ge 0.$$

Applying the Bellman-Gronwall inequality with above differential inequality, one has

$$s^2(x(t))$$
$$\le s^2(x(0))\cdot\exp\left[-4\left(x_{10}^2+x_{30}^2\right)t\right],\quad \forall\, t\ge 0,$$

this implies

$$\left|s(x(t))\right|$$
$$\le \left|s(x(0))\right|\cdot\exp\left[-2\left(x_{10}^2+x_{30}^2\right)t\right],\quad \forall\, t\ge 0,$$
$$\left|\sqrt{x_1^2(t)+x_3^2(t)}-\sqrt{a}\right|^2$$
$$\le \left|\sqrt{x_1^2(t)+x_3^2(t)}-\sqrt{a}\right|\cdot\left|\sqrt{x_1^2(t)+x_3^2(t)}+\sqrt{a}\right|$$
$$=\left|x_1^2(t)+x_3^2(t)-a\right|$$
$$=\left|s(x(t))\right|$$
$$\le \left|s(x(0))\right|\cdot\exp\left[-2\left(x_{10}^2+x_{30}^2\right)t\right],\quad \forall\, t\ge 0.$$

It yields

$$\left|\sqrt{x_1^2(t)+x_3^2(t)}-\sqrt{a}\right|$$
$$\le \sqrt{\left|s(x(0))\right|}\cdot\exp\left[-\left(x_{10}^2+x_{30}^2\right)t\right],\quad \forall\, t\ge 0. \tag{6}$$

Consequently, by (3) and (6), we conclude that

$$\left|x_1(t)-\sqrt{a}\cos\left[bt+\tan^{-1}\left(\frac{x_{30}}{x_{10}}\right)\right]\right|$$
$$=\left|\sqrt{x_1^2(t)+x_3^2(t)}\cos\left[bt+\tan^{-1}\left(\frac{x_{30}}{x_{10}}\right)\right]\right.$$
$$\left.-\sqrt{a}\cos\left[bt+\tan^{-1}\left(\frac{x_{30}}{x_{10}}\right)\right]\right|$$
$$=\left|\sqrt{x_1^2(t)+x_3^2(t)}-\sqrt{a}\right|\cdot\left|\cos\left[bt+\tan^{-1}\left(\frac{x_{30}}{x_{10}}\right)\right]\right|$$
$$\le\left|\sqrt{x_1^2(t)+x_3^2(t)}-\sqrt{a}\right|$$
$$\le \sqrt{\left|s(x(0))\right|}\cdot\exp\left[-\left(x_{10}^2+x_{30}^2\right)t\right],\quad \forall\, t\ge 0,$$

$$\left|x_3(t) - \sqrt{a}\sin\left[bt + \tan^{-1}\left(\frac{x_{30}}{x_{10}}\right)\right]\right|$$

$$= \left|\sqrt{x_1^2(t) + x_3^2(t)}\sin\left[bt + \tan^{-1}\left(\frac{x_{30}}{x_{10}}\right)\right] - \sqrt{a}\sin\left[bt + \tan^{-1}\left(\frac{x_{30}}{x_{10}}\right)\right]\right|$$

$$= \left|\sqrt{x_1^2(t) + x_3^2(t)} - \sqrt{a}\right| \cdot \left|\sin\left[bt + \tan^{-1}\left(\frac{x_{30}}{x_{10}}\right)\right]\right|$$

$$\le \left|\sqrt{x_1^2(t) + x_3^2(t)} - \sqrt{a}\right|$$

$$\le \sqrt{|s(x(0))|} \cdot \exp\left[-\left(x_{10}^2 + x_{30}^2\right)t\right], \quad \forall\, t \ge 0.$$

This completes the proof. □

Remark 1. It should be pointed out that, by Theorem 1, the system (1) can be regarded as nonlinear oscillators with the amplitude $\sqrt{a}$ and the frequency b. Such an oscillation is entirely independent of the initial condition and limit cycles of such an oscillation are not affected by parameter variation.

3. NUMERICAL SIMULATIONS

Example 1: Consider the system (1) with $(a, b) = (2, 4)$ and $x(0) = [2, 0, 2]^T$. By Theorem 1, we conclude that the phase trajectories of such a system tend to the exponentially stable limit cycle $s(x) = x_1^2 + x_3^2 - 2 = 0$ in the $x_1 - x_3$ plane, with the guaranteed convergence rate $\alpha = 4$. Besides, the states $x_1(t)$ and $x_3(t)$ exponentially track, respectively, the trajectories $\sqrt{2}\cos\left[4t + \frac{\pi}{4}\right]$ and $\sqrt{2}\sin\left[4t + \frac{\pi}{4}\right]$, in the time domain, with the guaranteed convergence rate $\frac{\alpha}{2} = 1$. Some state trajectories of such a system are depicted in Figure 1 and Figure 2.

Example 2: Consider the system (1) with $(a,b) = (3, 5)$ and $x(0) = [0.1, 0, -0.1]^T$. By Theorem 1, we conclude that the phase trajectories of such a system tend to the exponentially stable limit cycle $s(x) = x_1^2 + x_3^2 - 3 = 0$ in the $x_1 - x_3$ plane, with the guaranteed convergence rate $\alpha = 0.28$. In addition, the states $x_1(t)$ and $x_3(t)$ exponentially track, respectively, the trajectories $\sqrt{3}\cos\left[5t - \frac{\pi}{4}\right]$ and $\sqrt{3}\sin\left[5t - \frac{\pi}{4}\right]$, in the time domain, with the guaranteed convergence rate $\frac{\alpha}{2} = 0.14$. Some state trajectories of such a system are depicted in Figure 3 and Figure 4.

4. CONCLUSION

In this paper, the existence of limit cycles for a class of nonlinear systems has been considered. Based on the time-domain approach with differential and integral inequalities, the phenomenon of the stable limit cycle can be accurately verified for such nonlinear systems. The exponentially stable limit cycles, frequency of oscillation, and guaranteed convergence rate can also be correctly calculated. Finally, some numerical simulations have been given to demonstrate the feasibility and effectiveness of the main results.

ACKNOWLEDGEMENT

The author thanks the Ministry of Science and Technology of Republic of China for supporting this work under grants MOST 105-2221-E-214-025, MOST 106-2221-E-214-007, and MOST 106-2813-C-214-025-E.

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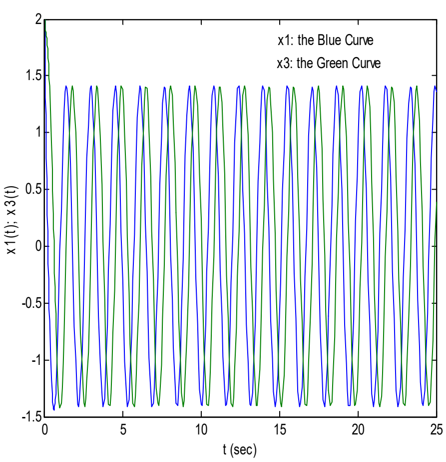

Figure 1: Typical state trajectories of the system (1) with $(a,b)=(2,4)$ and $x(0)=[2,0,2]^T$.

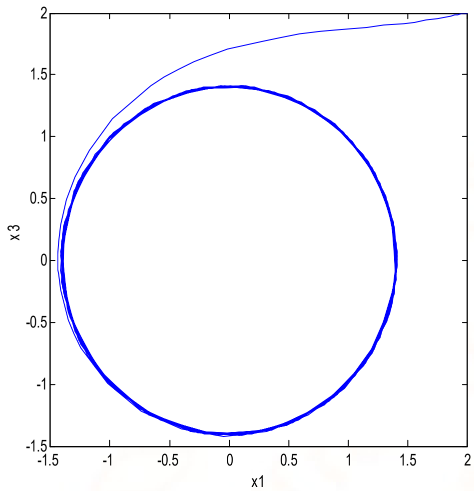

Figure 2: Typical phase trajectories of the system (1) with $(a,b)=(2,4)$ and $x(0)=[2,0,2]^T$.

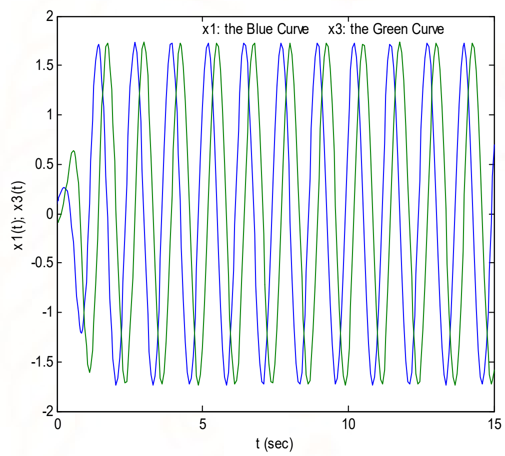

Figure 3: Typical state trajectories of the system (1) with $(a,b)=(3,5)$ and $x(0)=[0.1,0,-0.1]^T$.

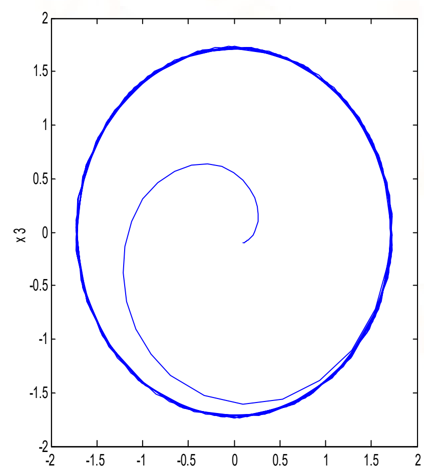

Figure 4: Typical phase trajectories of the system (1) with $(a,b)=(3,5)$ and $x(0)=[0.1,0,-0.1]^T$