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An Altering Distance Function in Fuzzy Metric Fixed Point Theorems

Dr C Vijender
Dept of Mathematics,
Sreenidhi Institute of Sciece and Technology,
Hyderabad, India

ABSTRACT

The aim of this paper is to improve conditions proposed in recently published fixed point results for complete and compact fuzzy metric spaces. For this purpose, the altering distance functions are used. Moreover, in some of the results presented the class of *t*-norms is extended by using the theory of countable extensions of *t*-norms.

Keywords: *Fixed point, Cauchy sequence, t-norm, altering distance, fuzzy metric space*

1. Introduction and Preliminaries:

In the fixed point theory in metric space an important place occupies the Banach contraction principle [1]. The mentioned theorem is generalized in metric spaces as well as in its various generalizations. In particular, the Banach contraction principle is observed in fuzzy metric spaces. There are several definitions of the fuzzy metric space [2–4]. George and Veeramani introduced the notion of a fuzzy metric space based on the theory of fuzzy sets, earlier introduced by Zadeh [5], and they obtained a Hausdorff topology for this type of fuzzy metric spaces [2, 3].

Recently, Shen *et al.* [6] introduced the notion of altering distance in fuzzy metric space (X,M,T) and by using the contraction condition

$$\varphi(M(fx, fy, t)) \le k(t)\cdot\varphi(M(x, y, t)),\ x, y\in X,\ x\neq y,\ t>0, \quad (1)$$

Obtained fixed point results for f: X→X. Using the same altering distance in this paper several fixed point results are proved in complete and compact fuzzy metric spaces introducing stronger contraction conditions than (1). Likewise, the contraction condition given in [7] is improved by using the altering distance, as well as by extending the class of *t*-norms.

First, the basic definitions and facts are reviewed.

Definition 1.1

A mapping $T:[0,1]\times[0,1]\to[0,1]$ is called a triangular norm (*t*-norm) if the following conditions are satisfied:

1. (T1) $T(a,1)=a$, $a\in[0,1]$,

2. (T2) $T(a, b)=T(b, a)$, $a, b\in[0,1]$,

3. (T3) $a\geq b, c\geq d\Rightarrow T(a, c)\geq T(b, d)$, $a, b, c, d\in[0,1]$,

4. (T4)$T(a,T(b,c))=T(T(a,b),c)$, $a,b,c\in[0,1]$.

Basic examples are $T_P(x, y)=x\cdot y$, $T_M(x, y)=\min\{x, y\}$, $T_L(x, y)=\max\{x+y-1,0\}$, and
$T_D(x, y)$ ={min(x, y),0,max(x, y)=1}, otherwise.T_D(x,

$$y) = \begin{cases} \min(x, y), \max(x, y) = 1, \\ 0, \text{otherwise} \end{cases}$$

Definition 1.2

A *t*-norm *T* is said to be positive if $T(a, b)>0$ whenever $a, b\in(0,1]$.

Definition 1.3

[2, 3] The 3-tuple (X, M, T) is said to be a fuzzy metric space if *X* is an arbitrary set, *T* is a continuous *t*-norm, and *M* is a fuzzy set on $X^2\times(0,\infty)$ such that the following conditions are satisfied:

(FM1) $M(x, y, t)>0$, $x, y\in X$, $t>0$,

(FM2) $M(x, y, t)=1$, $t>0\Leftrightarrow x=y$,

(FM3)$M(x, y, t)=M(y, x, t)$, $x, y\in X$, tt>0,

(FM4) $T(M(x, y, t),M(y, z, s))\leq M(x, z, t+s)$, $x, y, z\in X$, $t, s>0$,

(FM5)$M(x, y,\cdot):(0,\infty)\rightarrow[0,1]$ is continuous for every $x, y\in X$.

If (FM4) is replaced by condition
(FM4')$T(M(x, y, t),M(y, z, t))\leq M(x, z, t)$, $x, y, z\in X$, $t>0$,
then (X, M, T) is called a strong fuzzy metric space.

Moreover, if (X,M,T) is a fuzzy metric space, then *M* is a continuous function on $X\times X\times(0,\infty)$ and $M(x,y,\cdot)$ is non-decreasing for all $x, y\in X$.
If (X,M,T) is a fuzzy metric space, then *M* generates the Hausdorff topology on *X* (see [2, 3]) with a base of open sets $\{U(x, r, t):x\in X, r\in(0,1),t>0$, where $U(x, r, t)=\{y:y\in X,M(x, y, t)>1-r\}$.

Definition 1.4

[2, 3]Let (X, M, T) be a fuzzy metric space.

1. (a) A sequence $\{x_n\}_{n\in N}$ is a Cauchy sequence in (X, M, T) if for each $\varepsilon\in(0,1)$ and each $t>0$ there exist $n_0=n_0(\varepsilon, t)\in N$ such that $M(x_n, x_m, t)>1-\varepsilon$, for all $n, m\geq n_0$.

2. (b) A sequence $\{x_n\}_{n\in N}$ converges to *x* in (X, M, T) if for each $\varepsilon\in(0,1)$ and each t>0there exists $n_0=n_0(\varepsilon,t)\in N$ such that $M(x_n, x, t)>1-\varepsilon$ for all $n\geq n_0$. Then we say that $\{x_n\}_{n\in N}$ is convergent.

3. (c) A fuzzy metric space (X, M, T) is complete if every Cauchy sequence in (X, M, T) is convergent.

4. (d) A fuzzy metric space is compact if every sequence in *X* has a convergent subsequence.

It is well known [2] that in a fuzzy metric space every compact set is closed and bounded.

Definition 1.5

[8] Let T be a t-norm and $T_n:[0,1]\to[0,1]$, $n\in N$, be defined in the following way:

$T_1(x)=T(x, x), T_{n+1}(x)=T(T_n(x), x)$, $n\in N$, $x\in[0,1]$.

We say that the t-norm T is of H-type if the family $\{T_n(x)\}_{n\in N}$ is equi-continuous at $x=1$.

Each t-norm T can be extended in a unique way to an n-ary operation taking for $(x_1,\ldots,x_n)\in[0,1]^n$ the values $T_{i=1}^{0}x_i=1$, $T_{i=1}^{n}x_i=T(T_{i=1}^{n-1}x_i, x_n)$.

A t-norm T can be extended to a countable infinite operation taking for any sequence $(x_n)_{n\in N}$ from [0, 1] the value
$T_{i=1}^{\infty}x_i=\lim_{n\to\infty}T_{i=1}^{n}x_i$.

The sequence $(T_{i=1}^{n}x_i)_{n\in N}$ is non-increasing and bounded from below, hence the limit $T_{i=1}^{\infty}x_i$ exists.
In the fixed point theoryit is of interest to investigate the classes of t-norms T and sequences (x_n) from the interval [0, 1] such that $\lim_{n\to\infty}x_n=1$ and

$$\lim_{n\to\infty}T_{i=n}^{\infty}x_i=\lim_{n\to\infty}T_{i=n}^{\infty}x_{n+i}. \qquad (2)$$

It is obvious that

$$\lim_{n\to\infty}T_{i=n}^{\infty}x_i=1\Leftrightarrow\sum_{i=1}^{\infty}(1-x_i)<\infty \qquad (3)$$

for $T=T_L$ and $T=T_P$. (4)

For $T\geq T_L$ we have the following implication:

$$\lim_{n\to\infty}T_{i=n}^{\infty}x_i=1\Leftrightarrow\sum_{i=1}^{\infty}(1-x_i)<\infty. \qquad (5)$$

Proposition 1.7

Let $(x_n)_{n\in N}$ *be a sequence of numbers from* [0, 1] *such that* $\lim_{n\to\infty}x_n=1$ *and the t-norm T is of H-type. Then* $\lim_{n\to\infty}T_{i=n}^{\infty}x_i=\lim_{n\to\infty}T_{i=n}^{\infty}x_{n+i}=1$.

Theorem 1.8

Let (X, M, T) *be a fuzzy metric space, such that* $\lim_{t\to\infty}M(x, y, t)=1$. *Then if for some* $\sigma_0\in(0, 1)$ *and some* $x_0, y_0\in X$ *the following hold*:

$$\lim_{n\to\infty}T_{i=n}^{\infty}M\left(x_0, y_0, \frac{1}{\sigma_0^{\,i}}\right)=1,$$

then $\text{im}_{n\to\infty}T_{i=n}^{\infty}M\left(x_0, y_0, \frac{1}{\sigma^{i}}\right)=1$, *for every* $\sigma\in(0,1)$.

2. Main results:

A function $\varphi:[0,1]\rightarrow[0,1]$ is called an altering distance function if it satisfies the following properties:

1. (AD_1) φ is strictly decreasing and continuous;

2. (AD_2) $\varphi(\lambda)=0$ if and only if $\lambda=1$.

It is obvious that $\lim_{\lambda\rightarrow1^-}\varphi(\lambda)=\varphi(1)=0$.

Theorem 2.1

Let (X, M, T) *be a complete fuzzy metric space*, T *be a triangular norm and* $f:X\rightarrow X$. *If there exist* k_1, $k_2:(0,\infty)\rightarrow(0,1)$, *and an altering distance function* φ *such that the following condition*:
The following condition

$$\varphi(M(fx, fy, t)) \leq k_1(t)\cdot\min\{\varphi(M(x, y, t)), \varphi(M(x, fx, t)), \varphi(M(x, fy, 2t)), \varphi(M(y, fy, t))\} +k_2(t)\cdot\varphi(M(fx, y, 2t)),\ x, y\in X,\ x\neq y,\ t>0, \quad (6)$$

is satisfied, then f has a unique fixed point.

Proof

We observe a sequence $\{x_n\}$, where $x_0\in X$ and $x_{n+1}=fx_n$, $n\in N\cup\{0\}$. Note that, if there exists $n_0\in N\cup\{0\}$ such that $x_{n_0}=x_{n_0+1}=fx_{n_0}$, then x_{n_0} is a fixed point of f. Further, we assume that $x_n\neq x_{n+1}$, $n\in N_0$. Then

$$0<M(x_n, x_{n+1}, t)<1,\ t>0,\ n\in N_0. \quad (7)$$

If the pair $x=x_{n-1}$, $y=x_n$ satisfy condition (6) then

$$\varphi(M(x_n,x_{n+1},t))\leq k_1(t)\cdot\min\{\varphi(M(x_{n-1},x_n,t)),\varphi(M(x_{n-1},x_n,t)),\varphi(M(x_{n-1},x_{n+1},2t)),\varphi(M(x_n,x_{n+1},t))\} +k_2(t)\cdot\varphi(M(x_n,x_n,2t))$$

$$= k_1(t)\cdot\min\{\varphi(M(x_{n-1},x_n,t)),\varphi(M(x_{n-1},x_{n+1},2t)),\varphi(M(x_n,x_{n+1},t))\},n\in N,t>0. \quad (8)$$

By (FM_4) and (AD_1) we have

$$\varphi(M(x_{n-1},x_{n+1},2t))\leq\varphi(T(M(x_{n-1},x_n,t),M(x_n,x_{n+1},t))),\ n\in N,\ t>0,$$

and further

$$\varphi(M(x_n,x_{n+1},t))\leq k_1(t)\cdot\min\{\varphi(M(x_{n-1},x_n,t)),\varphi(T(M(x_{n-1},x_n,t),M(x_n,x_{n+1},t))),\varphi(M(x_n,x_{n+1},t))\},\quad n\in N,\ t>0. \quad (9)$$

Note that, by (T1), (T2), and (T3), it follows that

$$a=T(a,1)\geq T(a, b) \text{ and } b=T(b,1)\geq T(a, b)$$

and

$$\min\{\varphi(a), \varphi(b), \varphi(T(a, b))\}=\min\{\varphi(a), \varphi(b)\},\ a, b\in[0, 1].$$

Then by (9) we have

$$\varphi(M(x_n,x_{n+1},t))\leq k_1(t)\cdot\min\{\varphi(M(x_{n-1},x_n,t)),\varphi(M(x_n,x_{n+1},t))\},\ n\in N,\ t>0. \quad (10)$$

If we suppose that

$$\min\{\varphi(M(x_{n-1},x_n,t)),\varphi(M(x_n,x_{n+1},t))\}=\varphi(M(x_n,x_{n+1},t)),\ n\in N,\ t>0,$$

then

$$\varphi(M(x_n,x_{n+1},t))\leq k_1(t)\cdot\varphi(M(x_n,x_{n+1},t))<\varphi(M(x_n,x_{n+1},t)),\ n\in N,\ t>0.$$

So, by contradiction it follows that

$$\min\{\varphi(M(x_{n-1},x_n,t)),\varphi(M(x_n,x_{n+1},t))\}=\varphi(M(x_{n-1},x_n,t)),\ n\in N,\ t>0,$$

and by (10) we get

$\varphi(M(x_n,x_{n+1},t))\le k_1(t)\cdot\varphi(M(x_{n-1},x_n,t))<\varphi(M(x_{n-1},x_n,t))$, $n\in N$, $t>0$. (11)

By (AD_1) and it follows that the sequence $\{M(x_n,x_{n+1},t)\}$ is strictly increasing with respect to *n*, for every $t>0$. This fact, together with (7), implies that

$\lim_{n\to\infty}M(x_n,x_{n+1},t)=a(t)$, $a:(0,\infty)\to[0,1]$. (12)
But if we suppose that $a(t)\ne 1$, for some $t>0$, and let $n\to\infty$ in (11) we get a contradiction:

$\varphi(a(t))\le k_1(t)\cdot\varphi(a(t))<\varphi(a(t))$. (13)
So, $a\equiv 1$ in (12). Now we will show that the sequence $\{x_n\}$ is a Cauchy sequence. Suppose the contrary, *i.e.* that there exist $0<\varepsilon<10<\varepsilon<1$, $t>0$, and two sequences of integers $\{p(n)\}$, $\{q(n)\}$, $p(n)>q(n)>n$, $n\in N\cup\{0\}$, such that

$M(x_{p(n)},x_{q(n)},t)\le 1-\varepsilon$ and $M(x_{p(n)}-1,x_{q(n)},t)>1-\varepsilon$. (14)
By (12) for every ε_1, $0<\varepsilon_1<\varepsilon_0$, it is possible to find a positive integer n_1, such that for all $n>n_1$,

$M(x_{p(n)},x_{p(n)-1},t)\ge 1-\varepsilon_1$ and $M(x_{q(n)},x_{q(n)-1},t)\ge 1-\varepsilon_1$. (15)
Then we have

$M(x_{p(n)-1},x_{q(n)-1},t)\ge T(M(x_{p(n)-1},x_{q(n)},\frac{t}{2}),M(x_{q(n)},x_{q(n)-1},\frac{t}{2}))$, $n\in N$. (16)

Now using (14), (15), (16), and (T3) we have
$M(x_{p(n)-1},x_{q(n)-1},t)\ge T(1-\varepsilon,1-\varepsilon_1)$, $n>n_1$. (17)
Since ε_1 is arbitrary and *T* is continuous we have

$M(x_{p(n)-1},x_{q(n)-1},t)\ge T(1-\varepsilon,1)=1-\varepsilon$, $n>n_1$. (18)
Similarly, by (15) and (18)

$M(x_{p(n)},x_{q(n)-1},t)\ge 1-\varepsilon$ and $M(x_{p(n)},x_{q(n)},t)\ge 1-\varepsilon$, $n>n_1$. (19)
By (14) and (19) it follows that

$\lim_{n\to\infty}M(x_{p(n)},x_{q(n)},t)=1-\varepsilon$. (20)
If the pair $x=x_{p(n)-1}$, $y=x_{q(n)-1}$ satisfy condition (6) then
$\varphi(M(x_{p(n)},x_{q(n)},t))\le k_1(t)\cdot\min\{\varphi(M(x_{p(n)-1},x_{q(n)-1},t)),\varphi(M(x_{p(n)-1},x_{p(n)},t)),\varphi(M(x_{p(n)-1},x_{q(n)},2t)),\varphi(M(x_{q(n)-1},x_{q(n)},t))\}+$

$k_2(t)\cdot\varphi(M(x_{p(n)},x_{q(n)-1},2t))$. (21)
Letting $n\to\infty$, by (12), (14), (19), and (20), we have

$\varphi(1-\varepsilon)\le k_1(t)\cdot\min\{\varphi(1-\varepsilon),\varphi(1),\varphi(1-\varepsilon),\varphi(1)\}+k_2(t)\cdot\varphi(1-\varepsilon)<\varphi(1-\varepsilon)$. (22)
This is a contradiction. So $\{x_n\}$ is a Cauchy sequence.

Since (X, M, T) is a complete fuzzy metric space there exists $x\in X$ such that $\lim_{n\to\infty}x_n=x$. Let us prove, by contradiction, that *x* is fixed point for *f*. Suppose that $x\ne fx$.
If the pair $x=x$, $y=x_{n-1}$ satisfy condition (6) then
$\varphi(M(fx,x_n,t))\le k_1(t)\cdot\min\{\varphi(M(x,x_{n-1},t)),\varphi(M(x,fx,t)),\varphi(M(x,x_n,2t)),\varphi(M(x_{n-1},x_n,t))\}+k_2(t)\cdot\varphi(M(fx,x_{n-1},2t))$, $n\in N$, $t>0$. (23)

Letting $n\to\infty$ in (23) we have
$\varphi(M(fx,x,t))\le k_1(t)\cdot\min\{\varphi(M(x,x,t)),\varphi(M(x,fx,t)),\varphi(M(x,x,2t))\}+k_2(t)\cdot\varphi(M(fx,x,2t))<\varphi(M(fx,x,2t))<\varphi(M(fx,x,t))$, $t>0$.

So, by contradiction we conclude that $x=fx$.

Assume now that there exists another fixed point *v*, v≠x. Then applying (6) we have
$\varphi(M(x,v,t))\leq k_1(t)\cdot\min\{\varphi(M(x,v,t)),\varphi(M(x,x,t)),\varphi(M(x,v,2t)),\varphi(M(v,v,t))\}$

$+k_2(t)\cdot\varphi(M(x,v,2t))<\varphi(M(x,v,2t))<\varphi(M(x,v,t)),t>0.$ (24)
So, we get a contradiction, and *x* is a unique fixed point of the function *f*.

Theorem 2.2

Let (X, M, T) *be a complete fuzzy metric space and* f:X→X. *If there exist*
$k_1,k_2:(0,\infty)\to[0,1)$, $k_3:(0,\infty)\to(0,1)$, $\sum_{i=1}^{3} k_i(t)<1$, *and an altering distance functionφsuch that the following condition is satisfied*:
$\varphi(M(fx,fy,t))\leq k_1(t)\cdot\varphi(M(x,fx,t))+k_2(t)\cdot\varphi(M(y,fy,t))+k_3(t)\cdot\varphi(M(x,y,t))$ (25)
for all x, y∈X, x≠y, *and* t>0, *then f has a unique fixed point*.

Proof

Let $x_0\in X$ and $x_{n+1}=fx_n$. Suppose that $x_n\neq x_{n+1}$, $n\in N_0$, *i.e.*
$0<M(x_n,x_{n+1},t)<1,\ n\in N_0,\ t>0.$ (26)
By (25), with $x=x_{n-1}$, $y=x_n$, we have
$\varphi(M(x_n,x_{n+1},t))\leq k_1(t)\cdot\varphi(M(x_{n-1},x_n,t))+k_2(t)\cdot\varphi(M(x_n,x_{n+1},t))+k_3(t)\cdot\varphi(M(x_{n-1},x_n,t)),\ n\in N_0,\ t>0,$
(27)
i.e.
$\varphi(M(x_n,x_{n+1},t))\leq\frac{k_1(t)+k_3(t)}{1-k_2(t)}\varphi(M(x_{n-1},x_n,t)),\ n\in N_0,\ t>0.$ (28)
Since *φ* is strictly decreasing, the sequence $\{M(x_n, x_{n+1}, t)\}$ is strictly increasing sequence, with respect to *n*, for every t>0. Hence, by (26) and a similar method to Theorem 2.1 it could be shown that

$\lim_{n\to\infty}M(x_n, x_{n+1}, t)=1,\ t>0,$ (29)
and $\{x_n\}$ is a Cauchy sequence. So, there exists x∈X such that $\lim_{n\to\infty}x_n=x$. Now, by (25) with $x=x_{n-1}$, y=x we have

$\varphi(M(x_n,fx,t))\leq k_1(t)\cdot\varphi(M(x_{n-1},x_n,t))+k_2(t)\cdot\varphi(M(x,fx,t))+k_3(t)\cdot\varphi(M(xn-1,x,t)),$ (30)
n∈N, t>0. Letting n→∞ in (30) we have

$\varphi(M(x,fx,t))\leq k_1(t)\cdot\varphi(1)+k_2(t)\cdot\varphi(M(x,fx,t))+k_3(t)\cdot\varphi(1)$
$=k_2(t)\cdot\varphi(M(x,fx,t)),t>0,$ (31)
i.e.

$(1-k_2(t))\varphi(M(x,fx,t))\leq 0,\ t>0.$ (32)
It follows that φ(M(x, fx, t))=0 and x=fx.

Assume now that there exists a fixed point *v*, v≠x. Then by (25) we have

$\varphi(M(fx,fv,t))\leq k_1(t)\cdot\varphi(M(x,fx,t))+k_2(t)\cdot\varphi(M(v,fv,t))+k_3(t)\cdot\varphi(M(x,v,t))<\varphi(M(x,v,t)),t>0,$ (33)
which is a contradiction. So, *x* is a unique fixed point of *f*.

Remark 2.3

If in (25) we take $k_1(t)=k_2(t)=0$, t>0, we get condition (1), and Theorem 2.2. is a generalization of the result given in [24].

In the following theorems conditions (34) and (47) proposed in [32] are used to obtain fixed point results in complete and compact strong fuzzy metric spaces.

Theorem 2.4

Let (X, M, T)*be a complete strong fuzzy metric space with positive t-norm T and let* $f:X\to X$. *If there exists an altering distance function φ and* $a_i=a_i(t)$, $i=1,2,\dots,5$, $a_i>0$, $a_1+a_2+2a_3+2a_4+a_5<1$, *such that*

$$\varphi(T(r,s))\le\varphi(r)+\varphi(s), r,s\in\{M(x,fx,t):x\in X,t>0\} \tag{34}$$

and

$$\varphi(M(fx,fy,t))\le a_1\varphi(M(fx,x,t))+a_2\varphi(M(fy,y,t))+a_3\varphi(M(fx,y,t))+a_4\varphi(M(x,fy,t))+a_5\varphi(M(x,y,t)),\quad x,y\in X,\ t>0, \tag{35}$$

then f has a unique fixed point.

Proof

Let $x_0\in X$ be arbitrary. Define a sequence $(x_n)_{n\in N}$ such that $x_n=fx_{n-1}=f^n x_0$. By (35) with $x=x_{n-1}$ and $y=x_n$ we have

$$\varphi(M(x_n,x_{n+1},t))\le a_1\varphi(M(x_n,x_{n-1},t))+a_2\varphi(M(x_{n+1},x_n,t))+a_3\varphi(M(x_n,x_n,t))+a_4\varphi(M(x_{n-1},x_{n+1},t))+ a_5\varphi(M(x_{n-1},x_n,t)),\quad n\in N,\ t>0. \tag{36}$$

Since (X, M, T) is a strong fuzzy metric space we have
$M(x_{n-1},x_{n+1},t)\ge T(M(x_{n-1},x_n,t),M(x_n,x_{n+1},t))$, $n\in N$, $t>0$,

using (34) we obtained

$$\varphi(M(x_{n-1},x_{n+1},t))\le\varphi(T(M(x_{n-1},x_n,t),M(x_n,x_{n+1},t)))$$
$$\le\varphi(M(x_{n-1},x_n,t))+\varphi(M(x_n,x_{n+1},t)),\quad n\in N,\ t>0.$$

By (36) it follows that

$$\varphi(M(x_n,x_{n+1},t))\le\frac{a_1+a_4+a_5}{1-a_2-a_4}\varphi(M(x_{n-1},x_n,t))$$
$$<\varphi(M(x_{n-1},x_n,t)),\quad n\in N,\ t>0,$$

i.e.
$M(x_n,x_{n+1},t)>M(x_{n-1},x_n,t)$, $n\in N$, $t>0$.

So, the sequence $\{M(x_n,x_{n+1},t)\}$ is increasing and bounded and there exists
$\lim_{n\to\infty}M(x_n,x_{n+1},t)=p(t)$, $p:(0,\infty)\to[0,1]$.

Suppose that $p(t)\ne1$, for some $t>0$. Then, if we take $n\to\infty$ in (36)
$\varphi(p(t))\le(a_1+a_2+2a_4+a_5)\varphi(p(t))<\varphi(p(t))$,

and we get a contradiction, *i.e.*
$\lim_{n\to\infty}M(x_n,x_{n+1},t)=1$, $t>0$.

It remains to prove that $\{x_n\}_{n\in N}$ is a Cauchy sequence. Suppose the contrary, *i.e.* that there exist $\varepsilon>0$, $t>0$, such that for every $s\in N$ there exist $m(s)>n(s)\ge s$, and

$$M(x_{m(s)},x_{n(s)},t)<1-\varepsilon. \tag{37}$$

Let m(s) be the least integer exceeding n(s) satisfying the above property, *i.e.*

$M(x_{m(s)-1},x_{n(s)},t)\geq 1-\varepsilon$, $s\in N$, $t>0$. (38)

Then by (35) with $x=x_{m(s)-1}$ and $y=x_{n(s)-1}$, for each $s\in N$ and $t>0$ we have

$\varphi(M(x_{m(s)},x_{n(s)},t))\leq a_1\varphi(M(x_{m(s)},x_{m(s)-1},t))+a_2\varphi(M(x_{n(s)},x_{n(s)-1},t))+a_3\varphi(M(x_{m(s)},x_{n(s)-1},t))+$

$a_4\varphi(M(x_{m(s)-1},x_{n(s)},t))+a_5\varphi(M(x_{m(s)-1},x_{n(s)-1},t))$. (39)

By (FM4′), (34), and (AD_1) it follows that

$\varphi(M(x_{m(s)},x_{n(s)-1},t))\leq\varphi(M(x_{m(s)},x_{n(s)},t))+\varphi(M(x_{n(s)},x_{n(s)-1},t))$ (40)

and

$\varphi(M(x_{m(s)-1},x_{n(s)-1},t))\leq\varphi(M(x_{m(s)-1},x_{m(s)},t))+\varphi(M(x_{m(s)},x_{n(s)-1},t))$.

Combining the previous inequalities we get

$\varphi(M(x_{m(s)-1},x_{n(s)-1},t))\leq\varphi(M(x_{m(s)-1},x_{m(s)},t))+\varphi(M(x_{m(s)},x_{n(s)},t))+\varphi(M(x_{n(s)},x_{n(s)-1},t))$. (41)

Also, by (38) and (AD_1) we have

$\varphi(M(xm(s)-1,xn(s),t))\leq\varphi(1-\varepsilon)$. (42)

Inserting (40), (41), and (42) in (39) we obtain

$\varphi(M(x_{m(s)},x_{n(s)},t))\leq a_1\varphi(M(x_{m(s)},x_{m(s)-1},t))+a_2\varphi(M(x_{n(s)},x_{n(s)-1},t))+a_3\varphi(M(x_{m(s)},x_{n(s)},t))+a_3\varphi(M(x_{n(s)},x_{n(s)-1},t))+a_4\varphi(1-\varepsilon)+a_5\varphi(M(x_{m(s)},x_{m(s)-1},t))+a_5\varphi(M(x_{m(s)},x_{n(s)},t))+a_5\varphi(M(x_{n(s)},x_{n(s)-1},t))$,

i.e.

$(1-a_3-a_5)\varphi(M(x_{m(s)},x_{n(s)},t))\leq(a_1+a_5)\varphi(M(x_{m(s)},x_{m(s)-1},t))+(a_2+a_3+a_5)\varphi(M(x_{n(s)},x_{n(s)-1},t))+a_4\varphi(1-\varepsilon)$. (43)

By (37) it follows that

$\varphi(M(x_{m(s)},x_{n(s)},t))>\varphi(1-\varepsilon)$, (44)

and (43) and (44) imply

$(1-a_3-a_5)\varphi(1-\varepsilon)<(a_1+a_5)\varphi(M(x_{m(s)},x_{m(s)-1},t))+(a_2+a_3+a_5)\varphi(M(x_{n(s)},x_{n(s)-1},t))+a_4\varphi(1-\varepsilon)$. (45)

Letting $s\to\infty$ in (45) we have

$(1-a_3-a_5)\varphi(1-\varepsilon)\leq a_4\varphi(1-\varepsilon)$,

i.e.

$(1-a_3-a_4-a_5)\varphi(1-\varepsilon)\leq 0$,

which implies that $\varepsilon=0$ and we get a contradiction.

So, $\{x_n\}_{n\in N}$ is a Cauchy sequence and there exist $z\in X$ such that $\lim_{n\to\infty}x_n=z$. Now, by (35) with $x=x_{n-1}$ and $y=z$, we have

$\varphi(M(x_n,fz,t))\leq a_1\varphi(M(x_n,x_{n-1},t))+a_2\varphi(M(fz,z,t))+a_3\varphi(M(x_n,z,t))+a_4\varphi(M(x_{n-1},fz,t))+ a_5\varphi(M(x_{n-1},z,t))$, $n\in N$, $t>0$. (46)

Letting $n\to\infty$ in (46) we have

$(1-a_2-a_4)\varphi(M(z,fz,t))\leq 0$, $t>0$.

Therefore, $M(z, fz, t)=1$, $t>0$, and $z=fz$.

Suppose now that there exists another fixed point $w=fw$. By (35) with $x=z$ and $y=w$ we get

$(1-a_3-a_4-a_5)\varphi(M(z,w,t))\leq 0$, $t>0$,

i.e. $z=w$.

3. Conclusion:

In this paper several fixed point theorems in complete and compact fuzzy metric spaces are proved. For this purpose new contraction types of mappings with altering distances are proposed.

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