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Fixed Points in Continuous Non Decreasing Functions

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ABSTRACT

The purpose of this paper is to establish a common fixed point theorem is established in G-complete fuzzy metric spaces in the sense of George and Veeramani besides discussing some related problems.

Keywords— *Continuous t-norm, Fuzzy metric space, G- Cauchy sequence, Housdroff and first countable*

INTRODUCTION

Gregori and Sapena [1] introduced the notion of fuzzy contractive mapping and proved fixed point theorems in varied classes of complete fuzzy metric spaces in the senses of George and Veeramani [2], Kramosil and Michalek [3] and Grabiec's [4]. Soon after, Mihet [5] proposed a fuzzy fixed point theorem for (weak) Banach contraction in M-complete fuzzy metric spaces. In this continuation, Mihet [6,7] further enriched the fixed point theory for various contraction mappings in fuzzy metric spaces besides introducing variants of some new contraction mappings such as: Edelstein fuzzy contractive mappings, fuzzy $\psi-$contraction of (ϵ, λ) type etc. In the same spirit, Qiu et al. [8,9] also obtained some common fixed point theorems for fuzzy mappings under suitable conditions. In 2010 Pacurar and Rus in [10] , introduced the concept of cyclic ϕ-contraction and utilize the same to prove a fixed point theorem for cyclic $\phi-$contraction in the natural setting of complete metric spaces besides investigating several related problems in respect of fixed points.

In this paper the notion of cyclic weak $\phi-$contraction is introduced in a fuzzy metric space. Furthermore, fixed point theorem is established in G-complete fuzzy metric spaces in the sense of George and Veeramani besides discussing some related problems.

I. PRELIMINARIES

Definition − 1.1 A binary operation $\star: [0,1] \times [0,1] \rightarrow [0,1]$ is continuous $t - \text{norm}$ if $\star$ is satisfying the following conditions:

1.1 (i) $\star$ is commutative and associative.
1.1 (ii) $\star$ is continuous.
1.1 (iii) $a \star 1 = a$ for all $a \in [0,1]$.
1.1 (iv) $a \star b \leq c \star d$ whenever $a \leq c$ and $b \leq d$,
For $a, b, c, d \in [0,1]$.

Definition − 1.2 A triplet $(X, M, \star)$ is said to be a fuzzy metric space if X is an arbitrary set, $\star$ is a continuous $t-$norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following condition for all $x, y, z, s, t > 0$,

1.2 (FM − 1) $M(x, y, t) > 0$
1.2 (FM − 2) $M(x, y, t) = 1$ if and only if $x = y$.
1.2 (FM − 3) $M(x, y, t) = M(y, x, t)$

1.2 (FM − 4) $M(x, y, t) \star M(y, z, s) \leq M(x, z, t + s)$

1.2 (FM − 5) $M(x, y, \bullet) : (0, \infty) \to (0,1]$ is continuous.

Then M is called a fuzzy metric on X. The function $M(x, y, t)$ denote the degree of nearness between x and y with respect to t.

Example 1.3 Let (X, d) be a metric space. Define $a * b = \min\{a, b\}$ and

$$M(x, y, t) = \frac{t}{t + d(x, y)}$$

For all $x, y \in X$ and all $t > 0$. Then $(X, M, \star)$ is a Fuzzy metric space.
It is called the Fuzzy metric space induced by d.

We note that, $M(x, y, t)$ can be realized as the measure of nearness between x and y with respect to t. It is known that $M(x, y, \cdot)$ is non decreasing for all $x, y \in X$. Let $M(x, y, \star)$ be a fuzzy metric space for $t > 0$, the open ball

$$B(x, r, t) = \{y \in X: M(x, y, t) > 1 - r\}.$$

Now, the collection $\{B(x, r, t): x \in X, 0 < r < 1, t > 0\}$ is a neighborhood system for a topology τ on X induced by the fuzzy metric M. This topology is Housdroff and first countable.

Definition 1.5 A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, \star)$ is said to be a converges to x iff for each $\varepsilon > 0$ and each $t > 0$, $n_0 \in N$ such that $M(x_n, x, t) > 1 - \varepsilon$ for all $n \geq n_0$.

Definition 1.6 A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, \star)$ is said to be a G- Cauchy sequence converges to x iff for each $\varepsilon > 0$ and each $t > 0$, $n_0 \in N$ such that $M(x_m, x_n, t) > 1 - \varepsilon$ for all $m, n \geq n_0$.

II. MAIN RESULTS

Theorem − 2.1 Let $(X, M, \star)$ be a fuzzy metric spaces, $A_1, A_2, \ldots, A_m$ be closed subsets of X and $Y = \cup_{i=1}^{m} A_i$ be G − complete. Suppose that $\phi: [0, \infty) \to [0, \infty)$ is a continuous non decreasing function with $\phi(r) > 0$ for each $r \in (0, +\infty)$ and $\phi(0) = 0$. If $f: Y \to Y$ satisfying,

$$\left(\frac{1}{M^2(fx,fy,t)} - 1\right) \leq \left(\frac{1}{M^2(x,fx,t) \star M^2(y,fy,t)} - 1\right)$$
$$-\phi\left(\frac{1}{M^2(x,fx,t) \star M^2(y,fy,t)} - 1\right) \qquad 2.1(i)$$

then f has a unique fixed point $z \in \cap_{i=1}^{m} A_i$.

Proof : Let $x_0 \in Y = \cup_{i=1}^{m} A_i$ and set $x_n = fx_{n-1}$ $(n \geq 1)$. Clearly, we get $M(x_n, x_{n+1}, t) = M(fx_{n-1}, fx_n, t)$ for any $t > 0$. Beside for any $n \geq 0$, there exists $i_n \in \{1,2, \ldots, m\}$ such that $x_n \in A_{i_n}$ and $x_{n+1} \in A_{i_n+1}$. Then by 4.2.4(i), (for $t > 0$) we have

$$\left(\frac{1}{M^2(x_n,x_{n+1},t)} - 1\right) = \left(\frac{1}{M^2(fx_{n-1},fx_n,t)} - 1\right) \qquad 2.1(ii)$$
$$\leq \left(\frac{1}{M^2(x_{n-1},fx_{n-1},t) \star M^2(x_n,fx_n,t)} - 1\right)$$
$$-\phi\left(\frac{1}{M^2(x_{n-1},fx_{n-1},t) \star M^2(x_n,fx_n,t)} - 1\right)$$

$$\leq \left(\frac{1}{M^2(x_{n-1},x_n,t)\star M^2(x_n,x_{n+1},t)} - 1\right)$$

Which implies that

$$M^2(x_n, x_{n+1}, t) \geq M^2(x_{n-1}, x_n, t) \star M^2(x_n, x_{n+1}, t)$$

and hence

$$M(x_n, x_{n+1}, t) \geq M(x_{n-1}, x_n, t) \star M(x_n, x_{n+1}, t)$$

For all $n \geq 1$ and so $\{M(x_{n-1}, x_n, t)\}$ is non decreasing sequence of positive real numbers in $(0,1]$.

Let $S(t) = \lim_{n\to+\infty} M^2(x_{n-1}, x_n, t)$. Now we show that $S(t) = 1$ for all $t > 0$. If not, there exists some $t > 0$ such that $S(t) < 1$. Then, on making $n \to +\infty$ in4.2.4(ii), we obtain

$$\left(\frac{1}{S(t)} - 1\right) \leq \left(\frac{1}{S(t)} - 1\right) - \phi\left(\frac{1}{S(t)} - 1\right)$$

Which is a contradiction. Therefore $M^2(x_n, x_{n+1}, t) \to 1$ as $n \to +\infty$. Now for each positive integer p we have

$$M(x_n, x_{n+p}, t) \geq M\left(x_n, x_{n+1}, \frac{t}{p}\right) \star M\left(x_{n+1}, x_{n+2}, \frac{t}{p}\right)$$

$$\star \ldots \ldots \star M\left(x_{n+p-1}, x_{n+p}, \frac{t}{p}\right)$$

It follows that $\lim_{n\to+\infty} M(x_{n-1}, x_n, t) \geq 1 \star 1 \star \ldots\ldots \star 1 = 1$

So that $\{x_n\}$ is a G-Cauchy sequence. As Y is G-complete , then there exists $y \in Y$ such that $\lim_{n\to+\infty} x_n = y$. It follows that the iterative sequence $\{x_n\}$ has a infinite numbers of terms in A_i for each $i = 1,2, \ldots\ldots m$. Since Y is G-complete, for each A_i, $i = 1,2, \ldots\ldots m$ one can exists a subsequence of $\{x_n\}$ that converges to y. By virtue of the fact that each $A_i, i = 1,2, \ldots, m$ is closed, we conclude that $y \in \cap_{i=1}^{m} A_i \neq \phi$. Obviously, $\cap_{i=1}^{m} A_i$ is closed and $G-$ complete. Now we consider the restriction of f on $\cap_{i=1}^{m} A_i$, i.e. $f| \cap_{i=1}^{m} A_i : \cap_{i=1}^{m} A_i \to \cap_{i=1}^{m} A_i$ thus $f| \cap_{i=1}^{m} A_i$ has a unique fixed point in $\cap_{i=1}^{m} A_i$, say z, which is obtained by iteration form the starting point $x_0 \in Y$. To this end, we have show that $x_n \to z$ as $n \to \infty$, then, by 4.2.1(i), we have

$$\left(\frac{1}{M^2(x_n,z,t)} - 1\right) \leq \left(\frac{1}{M^2(x_{n-1},x_{n-2},t)} - 1\right) - \phi\left(\frac{1}{M^2(fz,z,t)} - 1\right)$$

Now letting $n \to +\infty$, we get

$$\left(\frac{1}{M^2(y,z,t)} - 1\right) \leq \left(\frac{1}{M^2(y,y,t)} - 1\right) - \phi\left(\frac{1}{M^2(z,z,t)} - 1\right)$$

Which is contradiction if $M(y, z, t) < 1$, and so, we conclude that $z = y$. Obviously z is the unique fixed point of f.

Theorem – 2.2 Let $(X, M, \star)$ be a fuzzy metric spaces, $A_1, A_2, \ldots, A_m$ be closed subsets of X and $Y = \cup_{i=1}^{m} A_i$ be $G-$ complete. Suppose that $\phi: [0, \infty) \to [0, \infty)$ is a continuous non decreasing function with $\phi(r) > 0$ for each $r \in (0, +\infty)$ and $\phi(0) = 0$. If $f: Y \to Y$ satisfying,

$$\left(\frac{1}{M^2(fx,fy,t)} - 1\right) \leq \left(\frac{1}{M^2(x,fx,t)\star M^2(y,fy,t)} - 1\right)$$

$$-\phi\left(\frac{1}{M^2(x,fx,t)\star M^2(y,fy,t)} - 1\right) \qquad 2.2(i)$$

and there exists a sequence $\{y_n\}$ in Y such that $M(y_n, fy_n, t) \to 1$ as $n \to +\infty$ for any $t > 0$, then $y_n \to z$ as $n \to +\infty$, provides that the fuzzy metric M is triangular, and z is the unique fixed point of f in $\cap_{i=1}^{m} A_i$.

Proof: By Theorem $-$ 4.2.4 we have that $z \in \cap_{i=1}^{m} A_i$ is the unique fixed point of f. Now, from the triangular inequality of M and 4.2.2(i), we have

$$\left(\frac{1}{M^2(y_n,z,t)} - 1\right) \leq \left(\frac{1}{M^2(y_n,fy_n,t)} - 1\right) + \left(\frac{1}{M^2(fy_n,fz,t)} - 1\right)$$

$$\left(\frac{1}{M^2(y_n,z,t)} - 1\right) \leq \left(\frac{1}{M^2(y_n,fy_n,t)} - 1\right)$$

$$+\left(\frac{1}{M^2(y_n,fy_n,t)\star M^2(fz,z,t)} - 1\right)$$

$$-\phi\left(\frac{1}{M^2(y_n,fy_n,t)\star M^2(fz,z,t)} - 1\right)$$

Which is equivalent to

$$\left(\frac{1}{M^2(y_n,z,t)} - 1\right) \leq \left(\frac{1}{M^2(y_n,fy_n,t)} - 1\right)$$

From the last inequality we conclude that

$$\lim_{n\to+\infty} \phi\left(\frac{1}{M^2(y_n,z,t)} - 1\right) = 0$$

Since $$\lim_{n\to+\infty} \left(\frac{1}{M^2(y_n,fy_n,t)} - 1\right) = 0\ .$$

then, by the property of ϕ, we conclude that $M(y_n, z, t) \to 1$, which is equivalent to say that $y_n \to z$ as $n \to +\infty$.

Theorem – 2.3 Let $(X, M, \star)$ be a fuzzy metric spaces, $A_1, A_2, \ldots, A_m$ be closed subsets of X and $Y = \cup_{i=1}^{m} A_i$ be G $-$ complete. Suppose that $\phi: [0, \infty) \to [0, \infty)$ is a continuous non decreasing function with $\phi(r) > 0$ for each $r \in (0, +\infty)$ and $\phi(0) = 0$. If $f: Y \to Y$ satisfying,

$$\left(\frac{1}{M^2(fx,fy,t)} - 1\right) \leq \left(\frac{1}{M^2(x,fx,t)\star M^2(y,fy,t)} - 1\right)$$

$$-\phi\left(\frac{1}{M^2(x,fx,t)\star M^2(y,fy,t)} - 1\right) \quad 2.3(i)$$

and there exists a sequence $\{y_n\}$ in Y such that $M(y_{n+1}, fy_n, t) \to 1$ as $n \to +\infty$ for any $t > 0$, then there exists $x \in Y$ such that $M(y_n, f^n x, t) \to 1$ as $n \to +\infty$ provides that the fuzzy metric M is triangular.

Proof : Once again, according to the proof of Theorem $-$ 4.2.4, we observe that for any initial point $x \in Y$, $z \in \cap_{i=1}^{m} A_i$ is the unique fixed point of f. Moreover, for any $t > 0$, $0 < M(y_n, z, t) < 1$ and $0 < M(y_{n+1}, z, t) < 1$. Set y as a limit of a convergent sequence $\{y_n\}$ in Y. Now, from the triangularity of M and 4.2.6(i), we have

$$\left(\frac{1}{M^2(y_{n+1},z,t)} - 1\right) \leq \left(\frac{1}{M^2(y_{n+1},fy_n,t)} - 1\right) + \left(\frac{1}{M^2(fy_n,fz,t)} - 1\right)$$

$$\left(\frac{1}{M^2(y_n,z,t)} - 1\right) \leq \left(\frac{1}{M^2(y_n,fy_n,t)} - 1\right) + \left(\frac{1}{M^2(y_n,fy_n,t)\star M^2(z,fz,t)} - 1\right) - \phi\left(\frac{1}{M^2(y_n,fy_n,t)\star M^2(z,fz,t)} - 1\right)$$

Then, on making $n \to +\infty$ in the above inequality, we get

$$\left(\frac{1}{M^2(y,z,t)} - 1\right) \leq \left(\frac{1}{M^2(y,fy,t)\star M^2(z,fz,t)} - 1\right) - \phi\left(\frac{1}{M^2(y,fy,t)\star M^2(z,fz,t)} - 1\right) \qquad 2.3(ii)$$

Clearly 4.2.6(ii) is true if and only if $\phi\left(\frac{1}{M^2(y,z,t)} - 1\right) = 0$. By the property of the function ϕ, must be $\left(\frac{1}{M^2(y,z,t)} - 1\right) = 0$ and thus $y = z$. Consequently, we have $M(y_n, f^n x, t) \to 1$ as $n \to +\infty$.

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